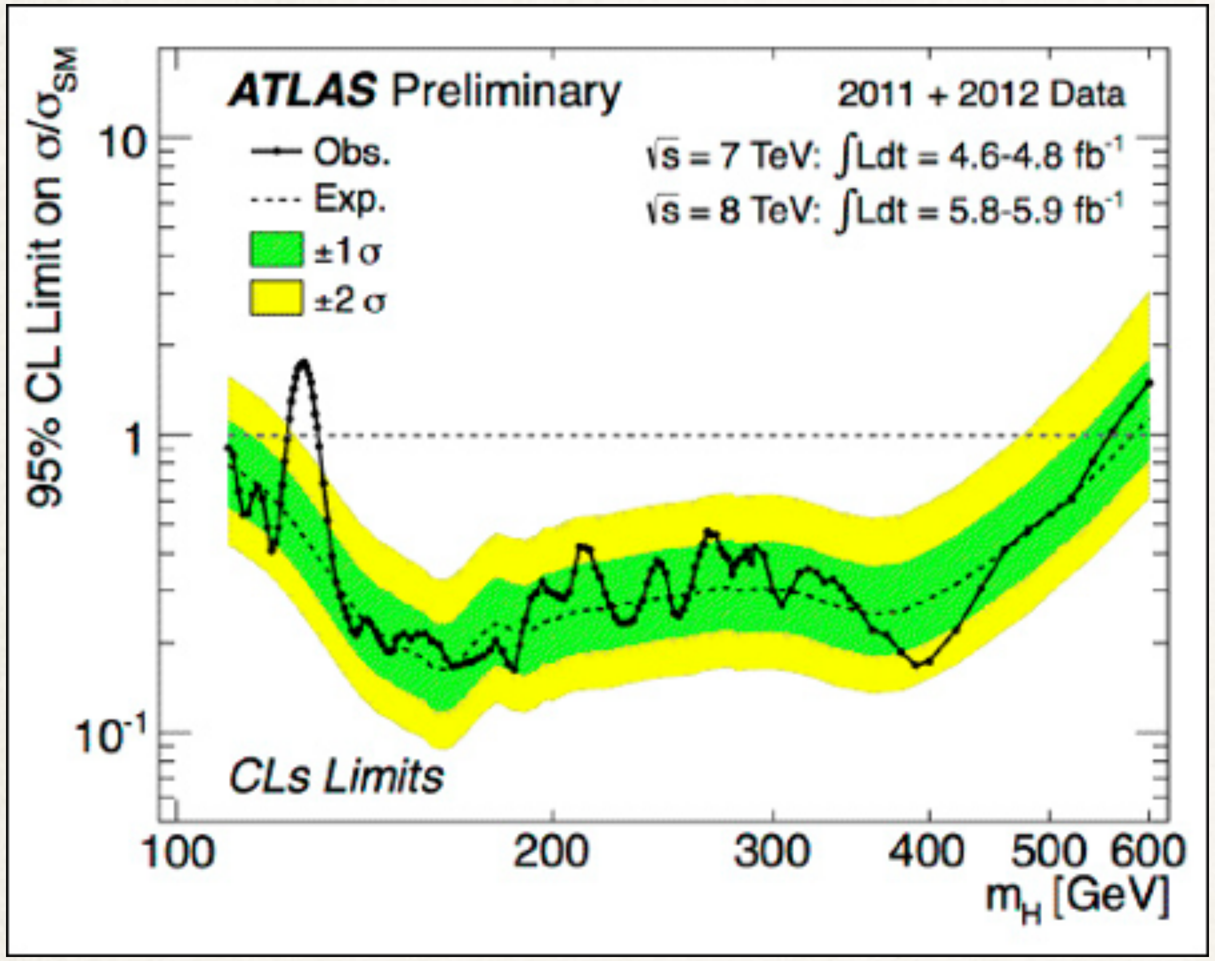
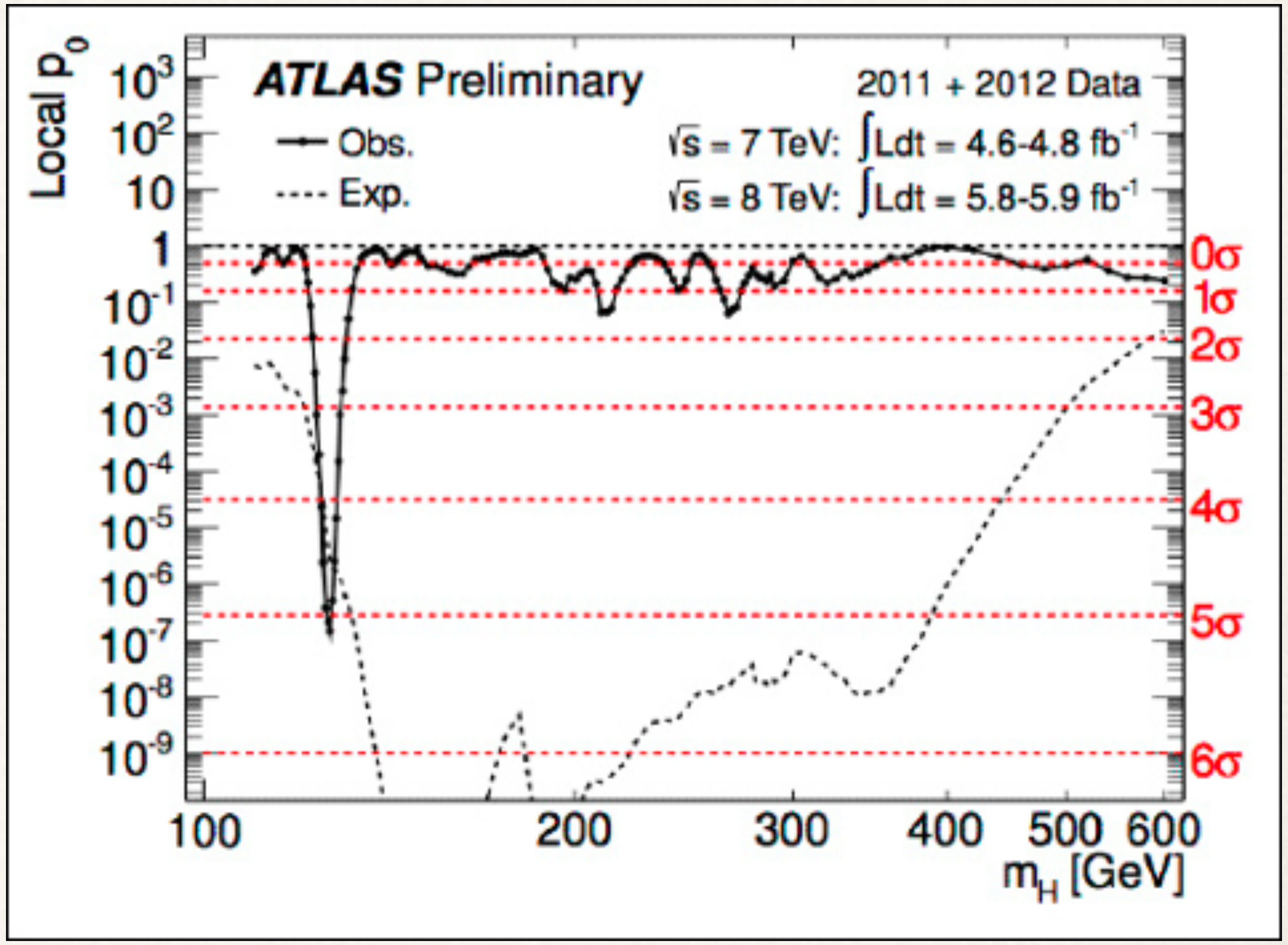




Basic statistics for HEP analysis

Youngjoon Kwon (Yonsei U.)

Jan. 13--17, 2014 The 10Th Saga-Yonsei Joint Workshop

apologies

- **freely taking from other people’s lecture slides, w/o properly citing the references**
    - just a rough list (from which I composed this lecture) is given
- **not paying attention to any mathematical rigor at all**
- **moreover, it will be simply impossible to cover “everything” within 70 minutes…**
    - so, I end up covering just a little fraction of the story…

References (*very rough*)

- Glen Cowan @ Cargese, July 2012
- Tom Junk @ TRIUMF, July 2009
- Bruce Yabsley @ BAS, Feb. 2011
- S. T'Jampens @ FAPPS '09, Oct. 2009
- mini-reviews on Probability & Statistics in RPP (PDG)
- ...

Outline

- **Basic elements**
    - some vocabulary
    - Probability axioms
    - some probability distributions
- **Two approaches: Freq. vs. Bayesian**
- **Hypothesis testing**
- **Parameter estimation**
- **Others**

Basic elements

some vocabulary

- **random variables, PDF, CDF**
- **expectation values**
- **mean, median, mode**
- **standard deviation, variance, covariance matrix**
- **correlation coefficients**
- **...**

Random variables and probability density functions

A random variable is a numerical characteristic assigned to an element of the sample space; can be discrete or continuous.

Suppose outcome of experiment is continuous value x

$$P(x \text{ found in } [x, x+dx]) = f(x)\,dx$$

→ $f(x)$ = probability density function (pdf) **PDF**

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

x must be somewhere

Or for discrete outcome x_i with e.g. i = 1, 2, ... we have

$$P(x_i) = p_i$$

probability mass function

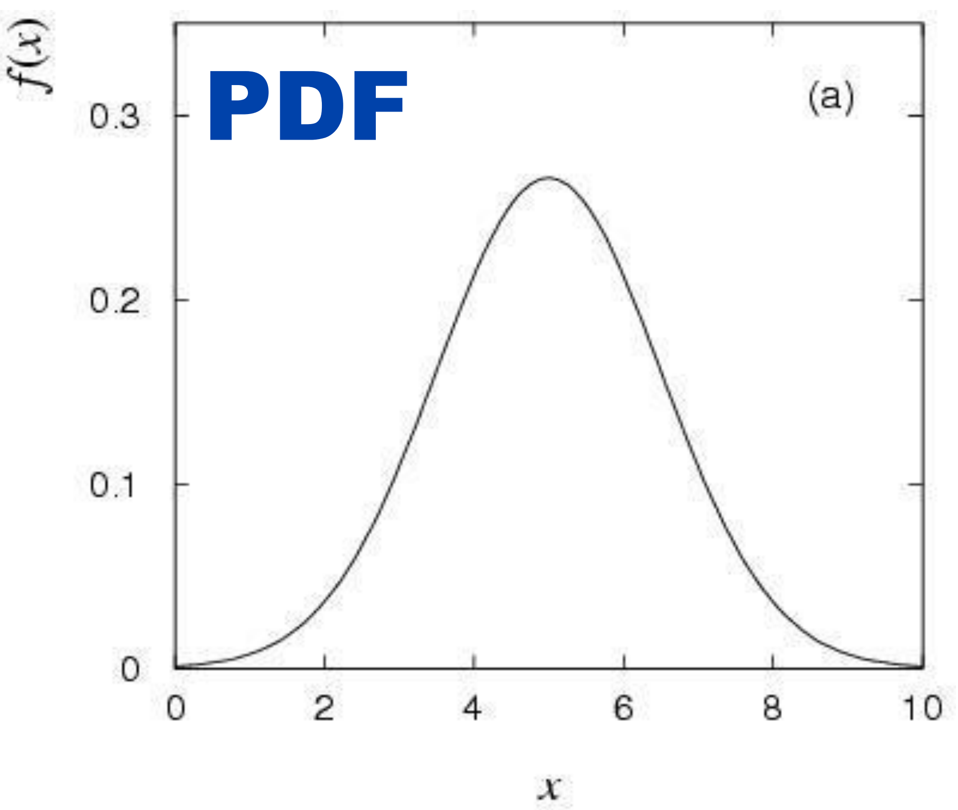
$$\sum_i P(x_i) = 1$$

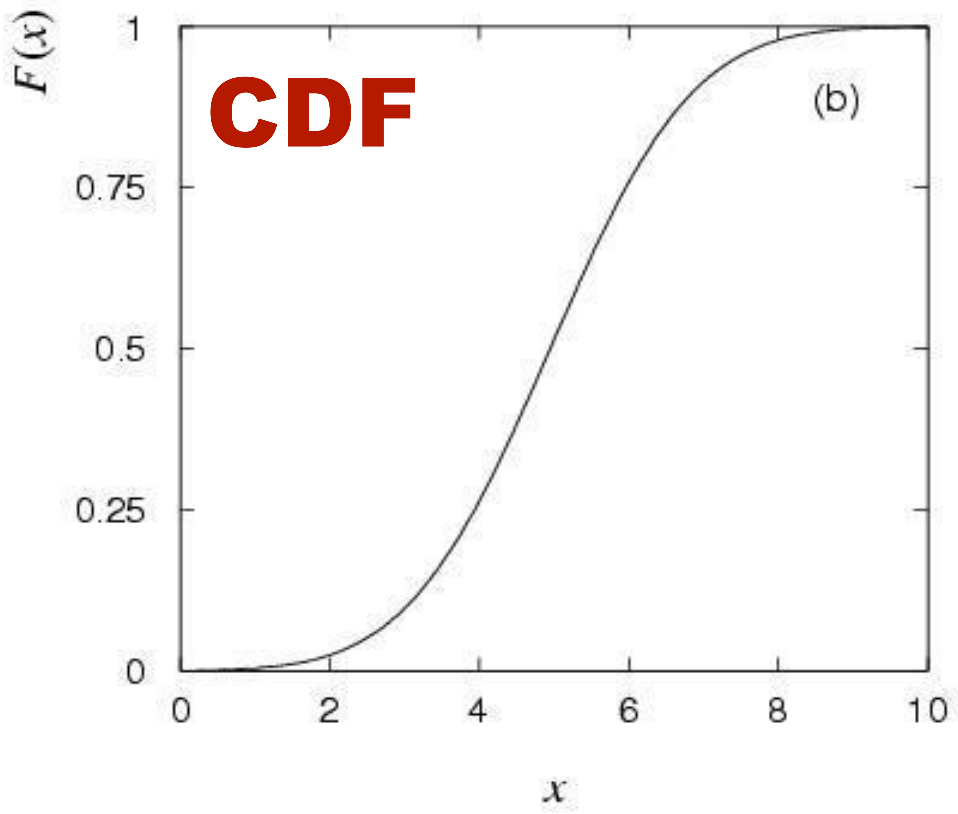
x must take on one of its possible values

Cumulative distribution function **CDF**

Probability to have outcome less than or equal to x is

$$\int_{-\infty}^{x} f(x')\,dx' \equiv F(x)$$ cumulative distribution function





Alternatively define pdf with $f(x) = \dfrac{\partial F(x)}{\partial x}$

Expectation: operator on f^{ns} of a random variable

discrete case: weighting by the probability

$$E(g) = \sum_{\Omega} P(X) \cdot g(X)$$

continuous case: integrating with p.d.f. as a weight

$$E(g) = \int_{\Omega} \mathrm{d}X\, f(X) g(X)$$

linear operator:

$$E[a \cdot g(X) + b \cdot h(X)] = a \cdot E[g(X)] + b \cdot E[h(X)]$$

We will rely on the linearity in what follows.

Expectations: mean, variance, covariance . . .

mean or expected value for the p.d.f. or *density* $f(X)$:

$$\mu = \overline{X} = \langle X \rangle = \int_\Omega \mathrm{d}X\, f(X) X = E(X)$$

variance for the p.d.f. (doesn't always exist!):

$$\begin{aligned}\sigma^2 = V(X) &= E\left[(X-\mu)^2\right] \\ &= E\left[X^2 - 2\mu X + \mu^2\right] \\ &= E(X^2) - \mu^2, \text{ which is more often written} \\ &= E(X^2) - [E(X)]^2 \\ &= \int_\Omega \mathrm{d}(X)\, f(X)(X-\mu)^2\end{aligned}$$

Note the mean and variance are specific to the *density* $f(X)$.
X itself is a random variable: what we focus on,
and think of as the underlying-true-situation, is $f(X)$

sample mean & sample variance

- n measurements $\{x_i\}$ where x_i follows $N(\mu, \sigma)$
- sample mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \ \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

With more measurements, the estimation of the mean will become more accurate.

- sample variance

$$V(x) = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2 = \overline{x^2} - \bar{x}^2$$

Sample variance approaches σ^2 for large n.

Expectations: mean, variance, covariance . . .

in multiple dimensions,

$$E\left[g(X,Y)\right] = \int\int_{\Omega} \mathrm{d}(X)\mathrm{d}(Y)\, f(X,Y) g(X,Y)$$

the mean is as before,

$$\mu_X = E(X) = \int\int_{\Omega} \mathrm{d}(X)\mathrm{d}(Y)\, f(X,Y) X$$

likewise the variance,

$$\sigma_X^2 = E\left[(X-\mu)^2\right] = \int\int_{\Omega} \mathrm{d}(X)\mathrm{d}(Y)\, f(X,Y)(X-\mu)^2$$

can now define the *covariance*,

$$\begin{aligned} \mathrm{cov}(X,Y) &= E\left[(X-\mu_X)(Y-\mu_Y)\right] \\ &= E(XY) - E(X)E(Y) \end{aligned}$$

Expectations: mean, variance, covariance . . .

more intuitive is the *correlation coefficient* given by

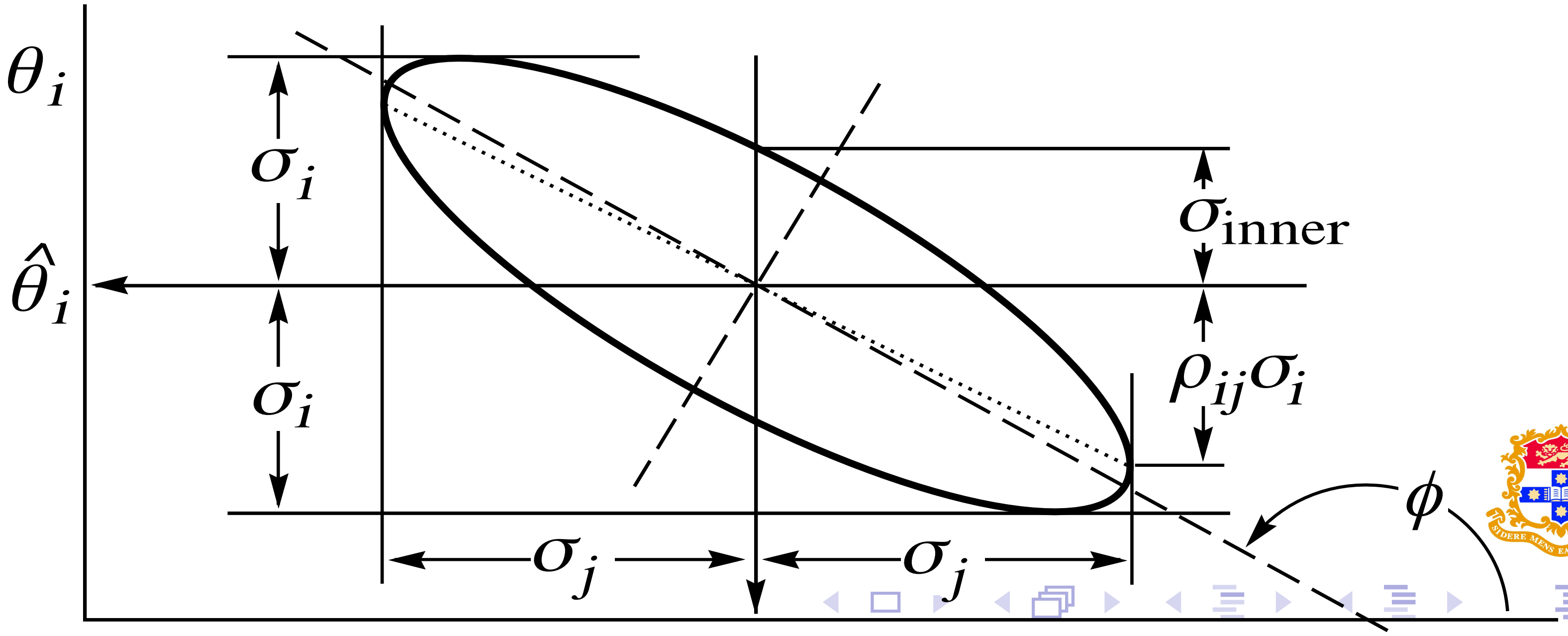
$$\rho(X, Y) = \frac{\mathrm{cov}(X, Y)}{\sigma_X \sigma_Y}$$

- This is bounded by one: $-1 \leq \rho(X, Y) \leq +1$
- For *independent* variables X, Y: $\rho(X, Y) = 0$
- But $\rho(X, Y) = 0 \not\Rightarrow X, Y$ independent (*e.g.* $Y = X^2$ case); remember independence is very difficult to arrange
- We have said nothing about Gaussians so far;
  we have said nothing about minimization so far
  — it is a property of a particular *density* $f(X, Y)$
- if the density is straightforward (unlike $Y = X^2$!!)
  there are great simplifications . . .

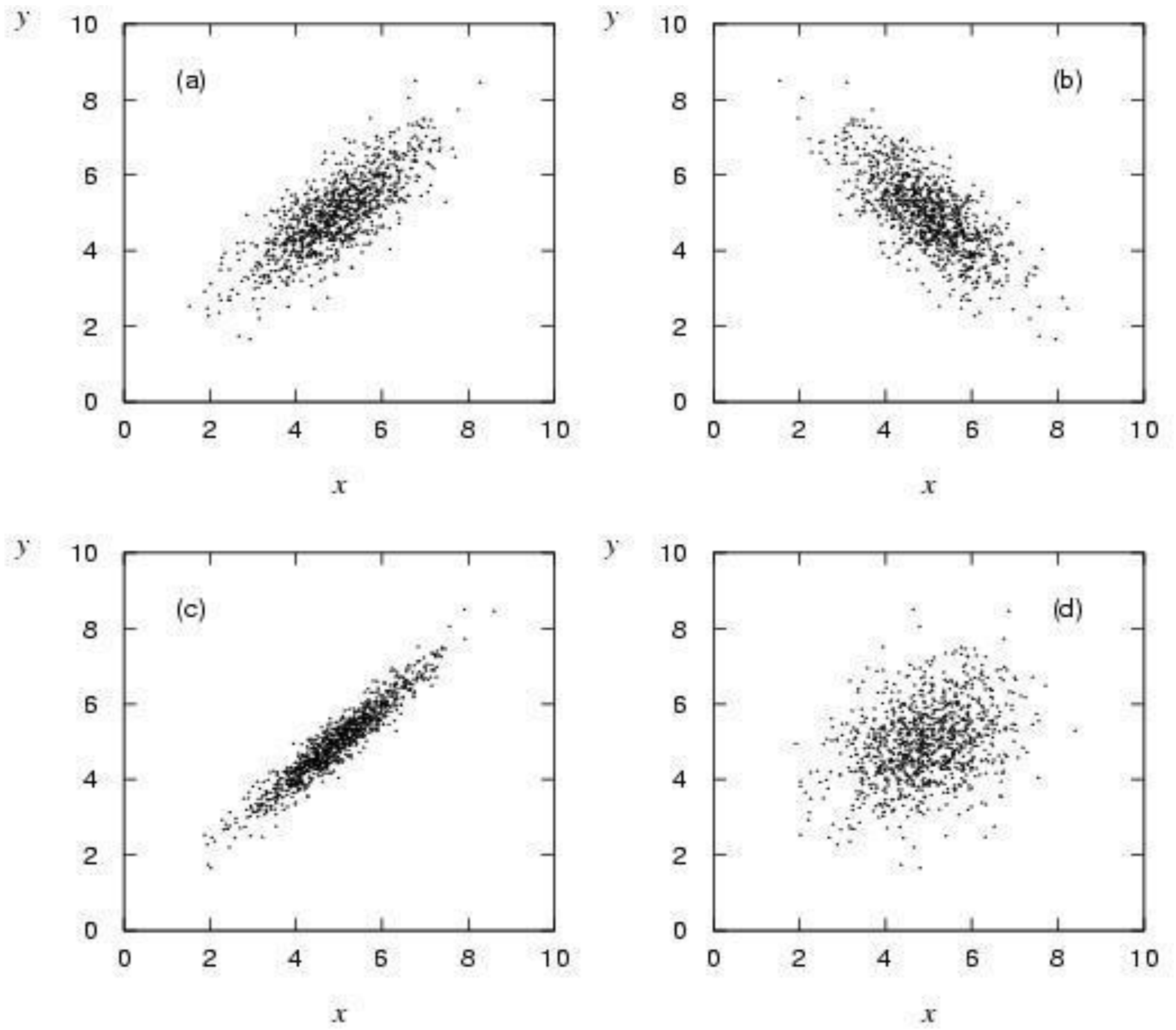
Expectations: covariance ... and fitting

- if $f(X_1, X_2, X_3, \ldots)$ is a multidimensional Gaussian, then $\mathrm{cov}(X_i, X_j)$ gives the *tilt* of the ellipsoid in (X_i, X_j)
- for $N \to \infty$, ML or weighted-least-squares fits return *parameter estimates* $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \ldots)$ distributed as a Gaussian about the *true* values θ underlying the data — frequentist interp^n: whole expt is a *single random throw*
- the covariances $\mathrm{cov}(\hat{\theta}_i, \hat{\theta}_j)$ form the *covariance matrix* or *error matrix*; the fitter *estimates it*
  - `HESSE`: from the second derivatives at $(\hat{\theta}_i, \hat{\theta}_j)$
  - `MINOS`: from the shape of $-2\ln\mathcal{L}$ about the minimum

$$\tan 2\phi = \frac{2\,\mathrm{cov}(\hat{\theta}_i, \hat{\theta}_j)}{\sigma_j^2 - \sigma_i^2} = \frac{2\rho_{ij}\sigma_i\sigma_j}{\sigma_j^2 - \sigma_i^2}$$



Correlation (cont.)



$\rho = 0.75$

$\rho = -0.75$

$\rho = 0.95$

$\rho = 0.25$

Error propagation on *f(x,y)*

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \left(\begin{array}{cc} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{array} \right) \left(\begin{array}{c} \partial f / \partial x \\ \partial f / \partial y \end{array} \right)$$

(Q) What if x and y are independent?

(HW) Obtain the error on $f(x,y) = C\,x\,y$

Statistics & Probability

Statistics is largely the inverse problem of probability.

- Probability:

  Know parameters of the theory ⇒ predict distributions of possible experimental outcomes

- Statistics:

  Know the outcome of an experiment ⇒ extract information about the parameters and/or the theory

  - Probability is the easier of the two – *more straightforward*.
  - Statistics is what we need as HEP analysts.
  - In HEP, the statistics issues often get very complex because we know so much about our data and need to incorporate all of what we find.

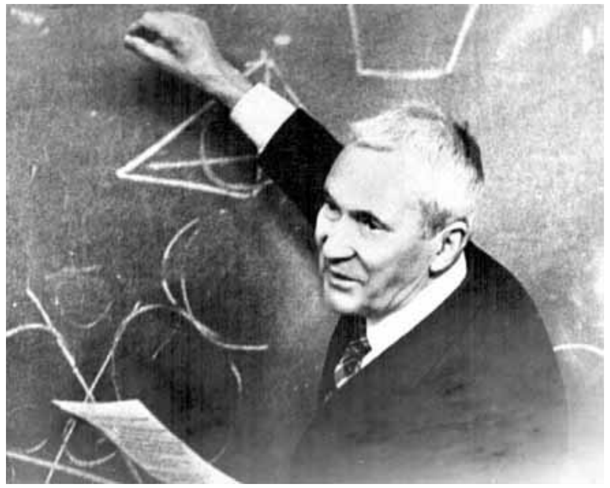
Probability Axioms

Consider a set S with subsets A, B, ...

For all $A \subset S, P(A) \geq 0$

$$P(S) = 1$$

If $A \cap B = \emptyset, P(A \cup B) = P(A) + P(B)$



Kolmogorov (1933)

Also define conditional probability:

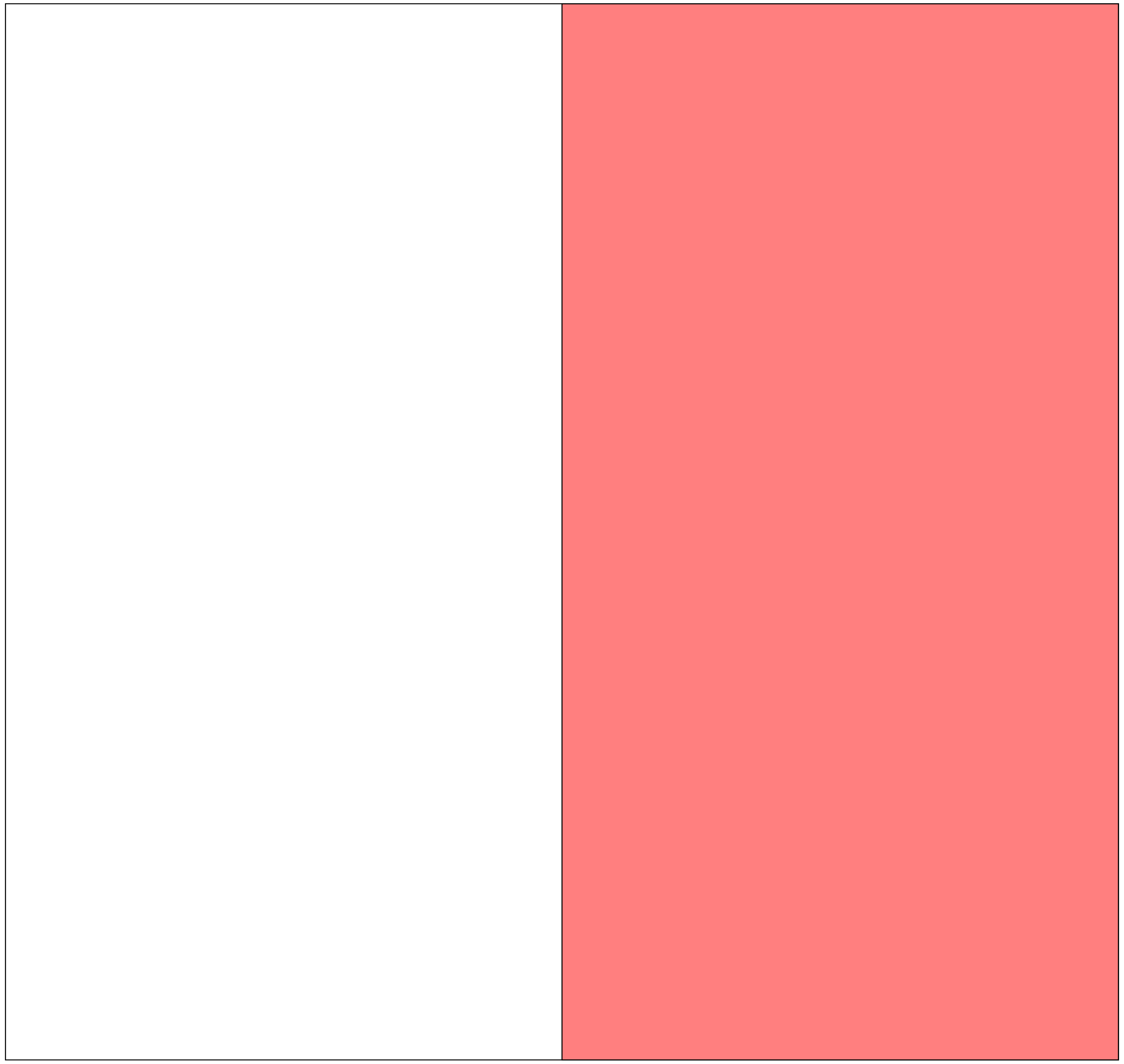
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Probability: $P(A|B) \neq P(B|A)$

An extreme (and personal) case:

- Ω : all people

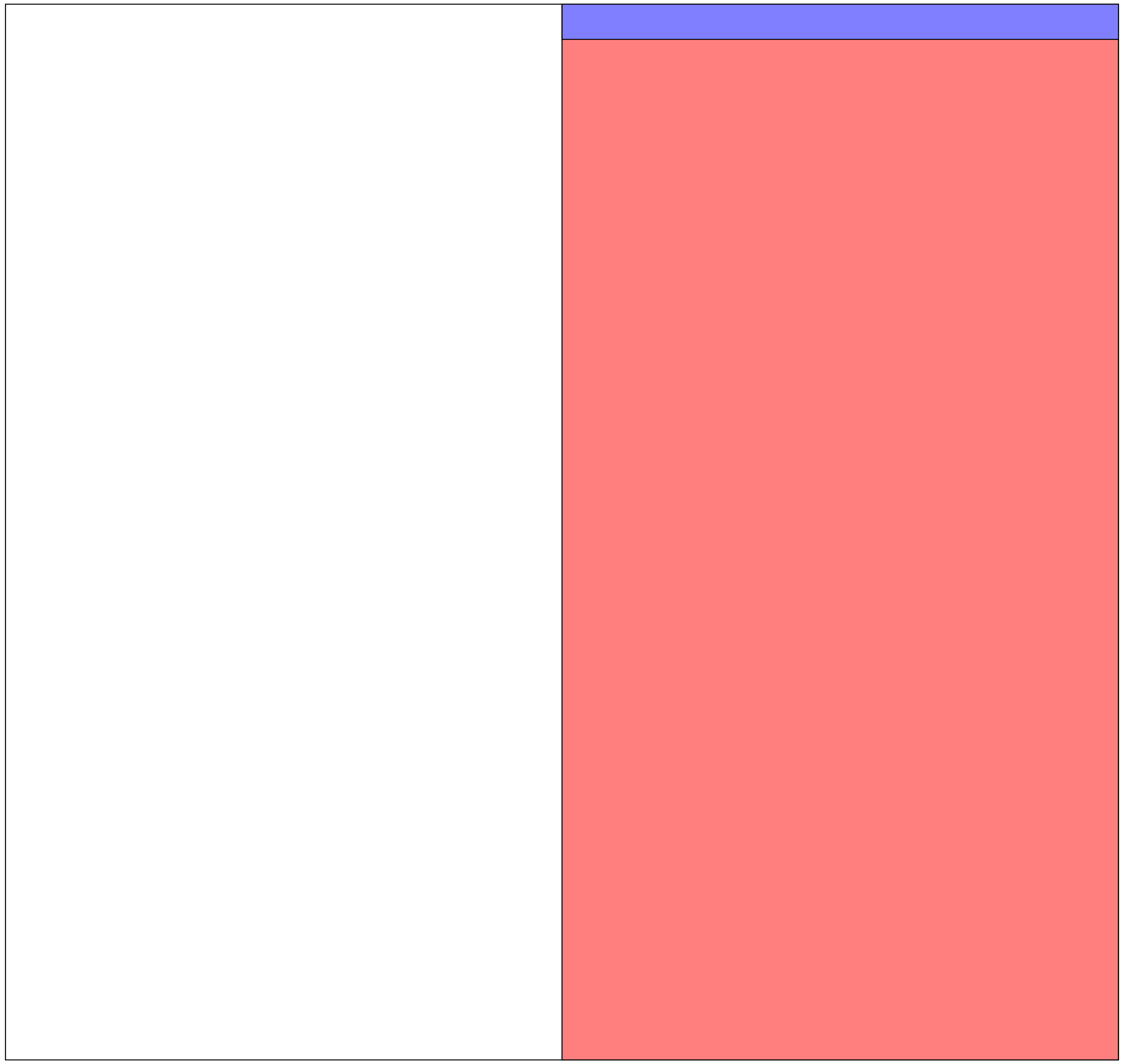
Probability: $P(A|B) \neq P(B|A)$



An extreme (and personal) case:

- Ω : all people
- $P(\text{woman}) = 50\%$

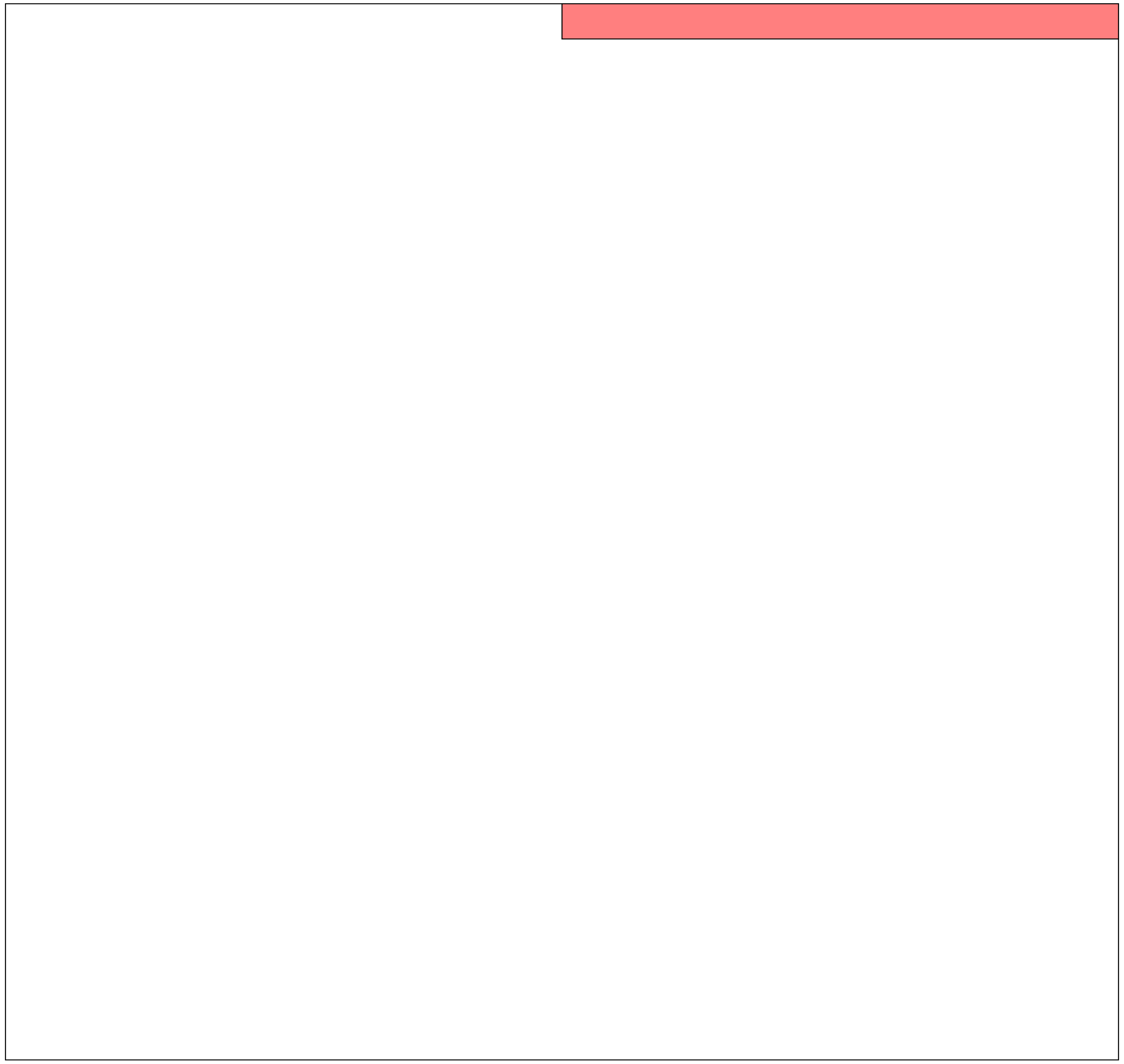
Probability: $P(A|B) \neq P(B|A)$



An extreme (and personal) case:

- Ω : all people
- $P(\text{woman}) = 50\%$
- $P(\text{pregnant} \mid \text{woman}) = 3\%$

Probability: $P(A|B) \neq P(B|A)$



An extreme (and personal) case:

- Ω : all people
- $P(\text{woman}) = 50\%$
- $P(\text{pregnant} \mid \text{woman}) = 3\%$
- $P(\text{pregnant}) = 1.5\%$

Probability: $P(A|B) \neq P(B|A)$

An extreme (and personal) case:

- Ω : all people
- $P(\text{woman}) = 50\%$
- $P(\text{pregnant} \mid \text{woman}) = 3\%$
- $P(\text{pregnant}) = 1.5\%$
- $P(\text{woman} \mid \text{pregnant}) = 100\%$

Indeed

$$P(w|p) = \frac{P(p|w) \cdot P(w)}{P(p)}$$

Two approaches

- **Relative frequency** **Frequentist**

A, B, ... are outcomes of a repeatable experiment

$$P(A) = \lim_{n\to\infty} \frac{\text{times outcome is } A}{n}$$

- **Subjective probability** **Bayesian**

A, B, ... are hypotheses (statements that are true or false)

$$P(A) = \text{degree of belief that } A \text{ is true}$$

Frequentist approach is, in general, easy to understand, but some HEP phenomena are best expressed by subjective prob., e.g. systematic uncertainties, prob(Higgs boson exists), ...

Measurement with errors

- **Let's say we are doing a single measurement**

$$x = a \pm b$$

- **Frequentist interpretation**
  - Repeating the measurement many times under identical conditions ("ensemble"), in 68.3% of those results, the true value of x will lie between a - b and $a + b$

- **Result of each measurement is a sampling from a Gaussian distribution with mean μ and width σ**
  - We may not know μ
  - We have some idea about σ -- experimental sensitivity

some useful distributions

Distribution	Probability density function f (variable; parameters)	Characteristic function $\phi(u)$	Mean	Variance σ^2
Uniform	$f(x;a,b) = \begin{cases} 1/(b-a) & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{ibu} - e^{iau}}{(b-a)iu}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$
Binomial	$f(r;N,p) = \dfrac{N!}{r!(N-r)!} \; p^r q^{N-r}$ $r = 0,1,2,\ldots,N \; ; \quad 0 \leq p \leq 1 \; ; \quad q = 1-p$	$(q + pe^{iu})^N$	Np	Npq
Poisson	$f(n;\nu) = \dfrac{\nu^n e^{-\nu}}{n!} \; ; \quad n = 0,1,2,\ldots \; ; \quad \nu > 0$	$\exp[\nu(e^{iu} - 1)]$	ν	ν
Normal (Gaussian)	$f(x;\mu,\sigma^2) = \dfrac{1}{\sigma\sqrt{2\pi}} \; \exp(-(x-\mu)^2/2\sigma^2)$ $-\infty < x < \infty \; ; \quad -\infty < \mu < \infty \; ; \quad \sigma > 0$	$\exp(i\mu u - \frac{1}{2}\sigma^2 u^2)$	μ	σ^2
Multivariate Gaussian	$f(\boldsymbol{x};\boldsymbol{\mu},V) = \dfrac{1}{(2\pi)^{n/2}\sqrt{\lvert V \rvert}}$ $\times \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T V^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]$ $-\infty < x_j < \infty; \quad -\infty < \mu_j < \infty; \quad \lvert V \rvert > 0$	$\exp\left[i\boldsymbol{\mu}\cdot\boldsymbol{u} - \frac{1}{2}\boldsymbol{u}^T V \boldsymbol{u}\right]$	$\boldsymbol{\mu}$	V_{jk}
χ^2	$f(z;n) = \dfrac{z^{n/2-1}e^{-z/2}}{2^{n/2}\Gamma(n/2)} \; ; \quad z \geq 0$	$(1-2iu)^{-n/2}$	n	$2n$

Binomial distribution

- **Given a repeated set of N trials, each of which has probability p of "success" (hence 1–p of "failure"), what is the distribution of the number of successes if the N trials are repeated over and over?**

$$\mathrm{Binom}(k \mid N, p) = \binom{N}{k} p^k (1-p)^{N-k}, \quad \sigma(k) = \sqrt{\mathrm{Var}(k)} = \sqrt{Np(1-p)}$$

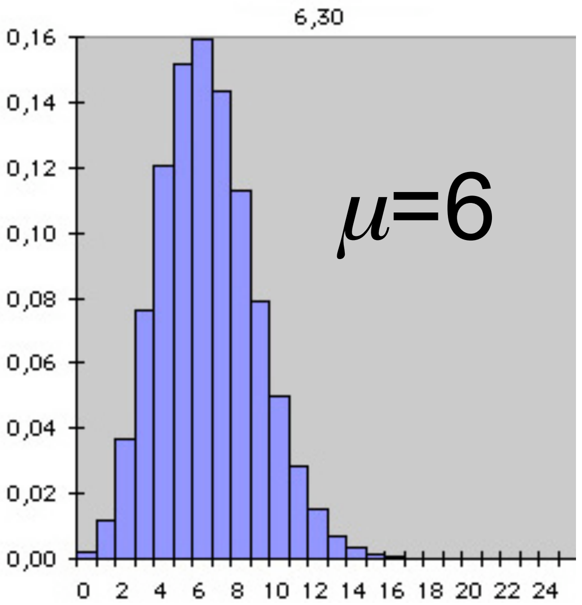
where k is the number of success trials

  - (Ex) events passing a selection cut, with a fixed total N

Poisson distribution

- Limit of Binomial when $N \to \infty$ and $p \to 0$ with $Np = \mu$ being finite and fixed $\Rightarrow$ Poisson distribution

$$\text{Poiss}(k \mid \mu) = \frac{e^{-\mu}\mu^{k}}{k!} \qquad \sigma(k) = \sqrt{\mu}$$



Normalized to
unit area in
two different senses

$$\sum_{k=0}^{\infty} \text{Poiss}(k \mid \mu) = 1, \quad \forall \mu$$

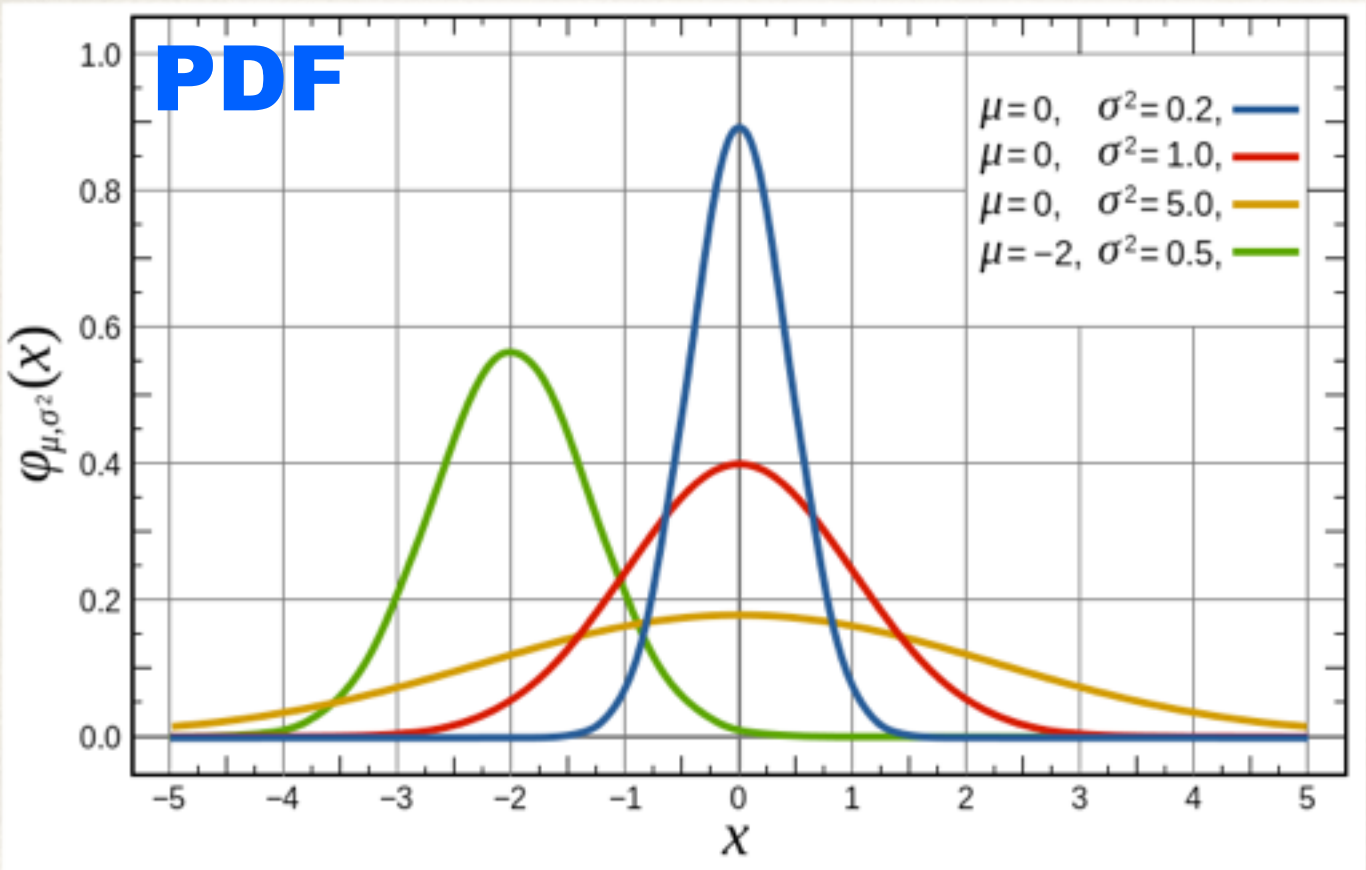
$$\int_0^{\infty} \text{Poiss}(k \mid \mu)\, d\mu = 1 \qquad \forall k$$

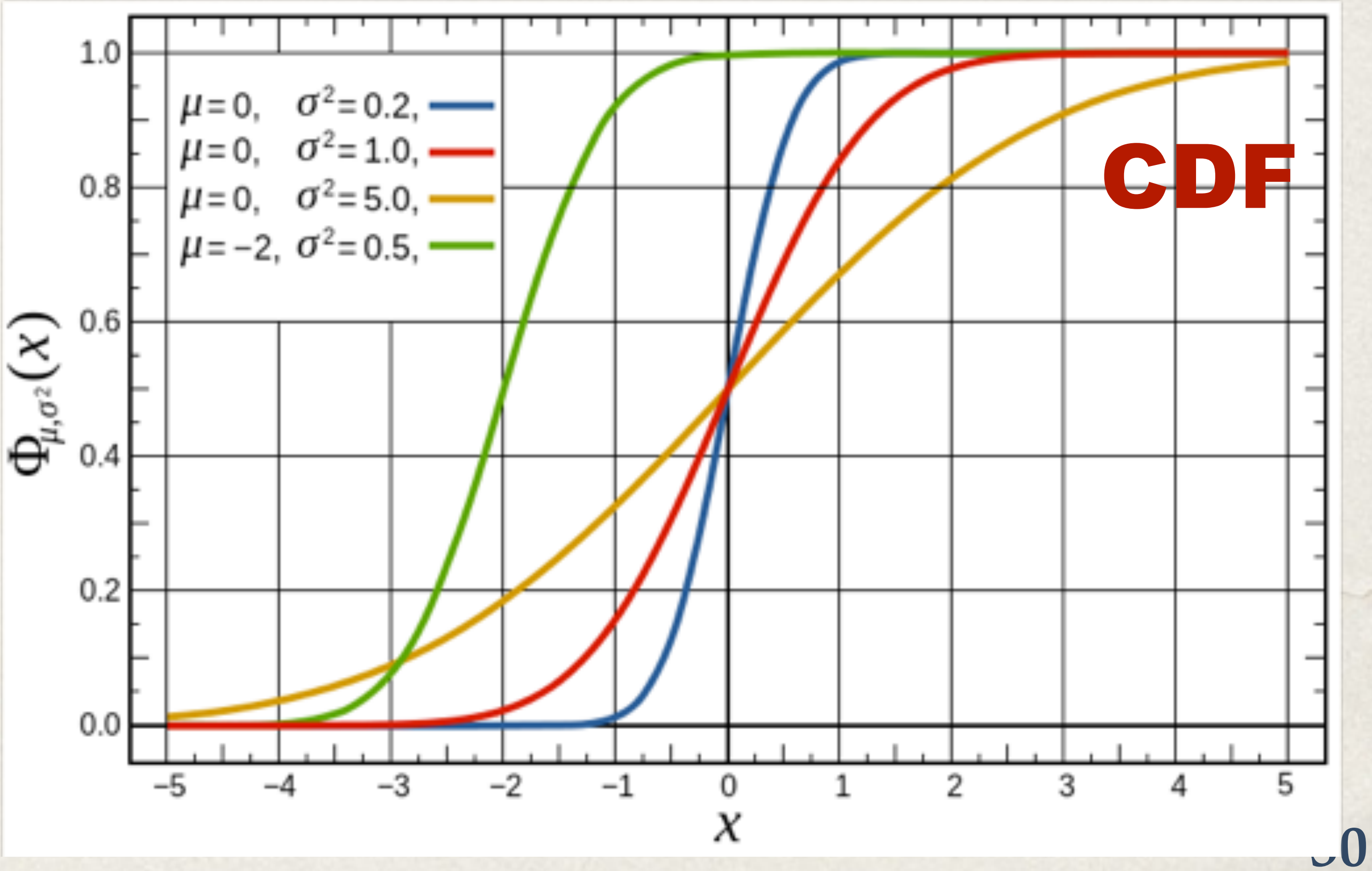
All counting results in HEP are assumed to be Poisson-distributed

Gaussian (Normal) distribution

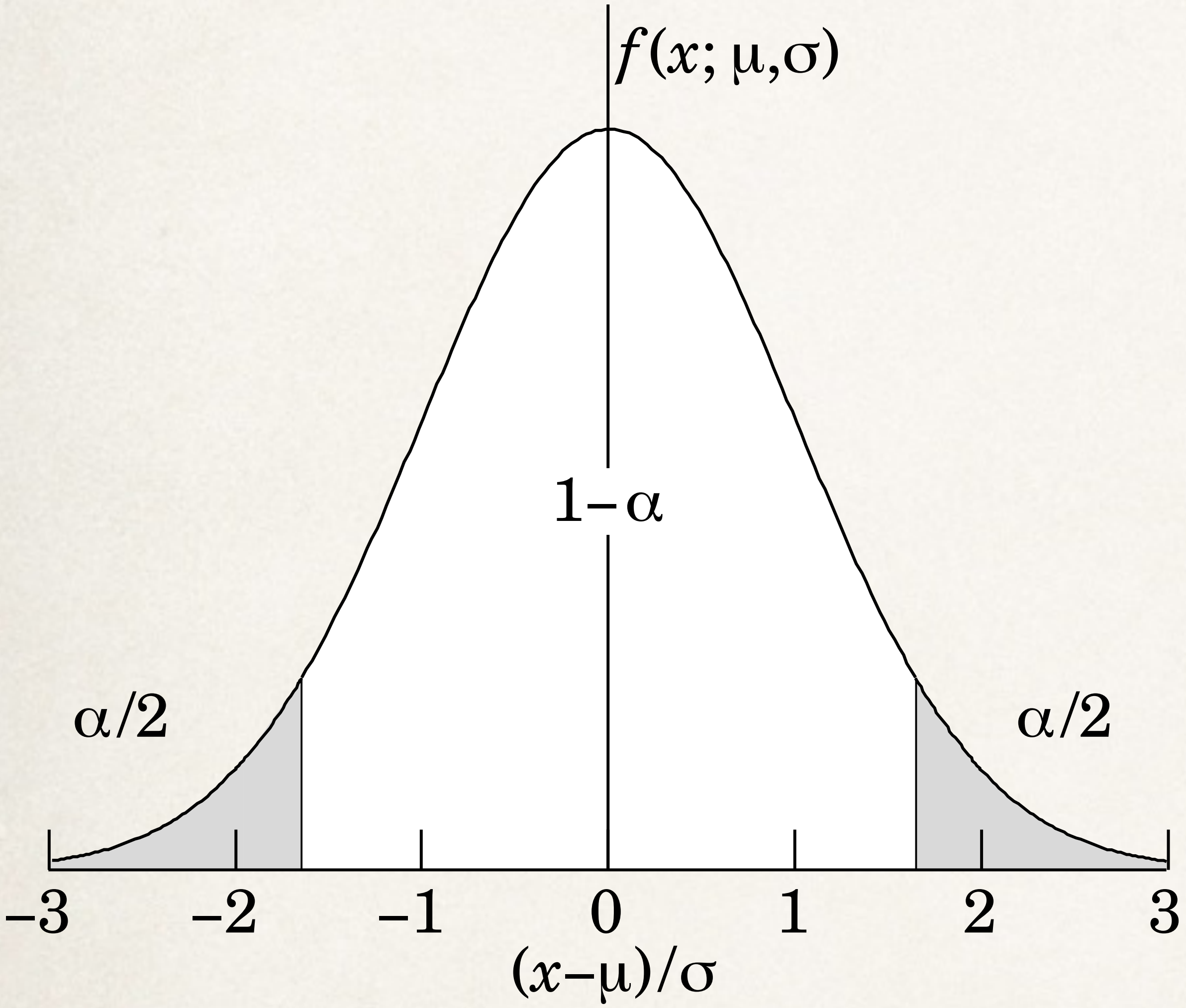
$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\ \exp(-(x-\mu)^2/2\sigma^2)$$

$$\int_{-\infty}^{x} f(x)dx = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-\mu}{\sqrt{2\sigma^2}}\right)\right]$$





Gaussian (Normal) distribution

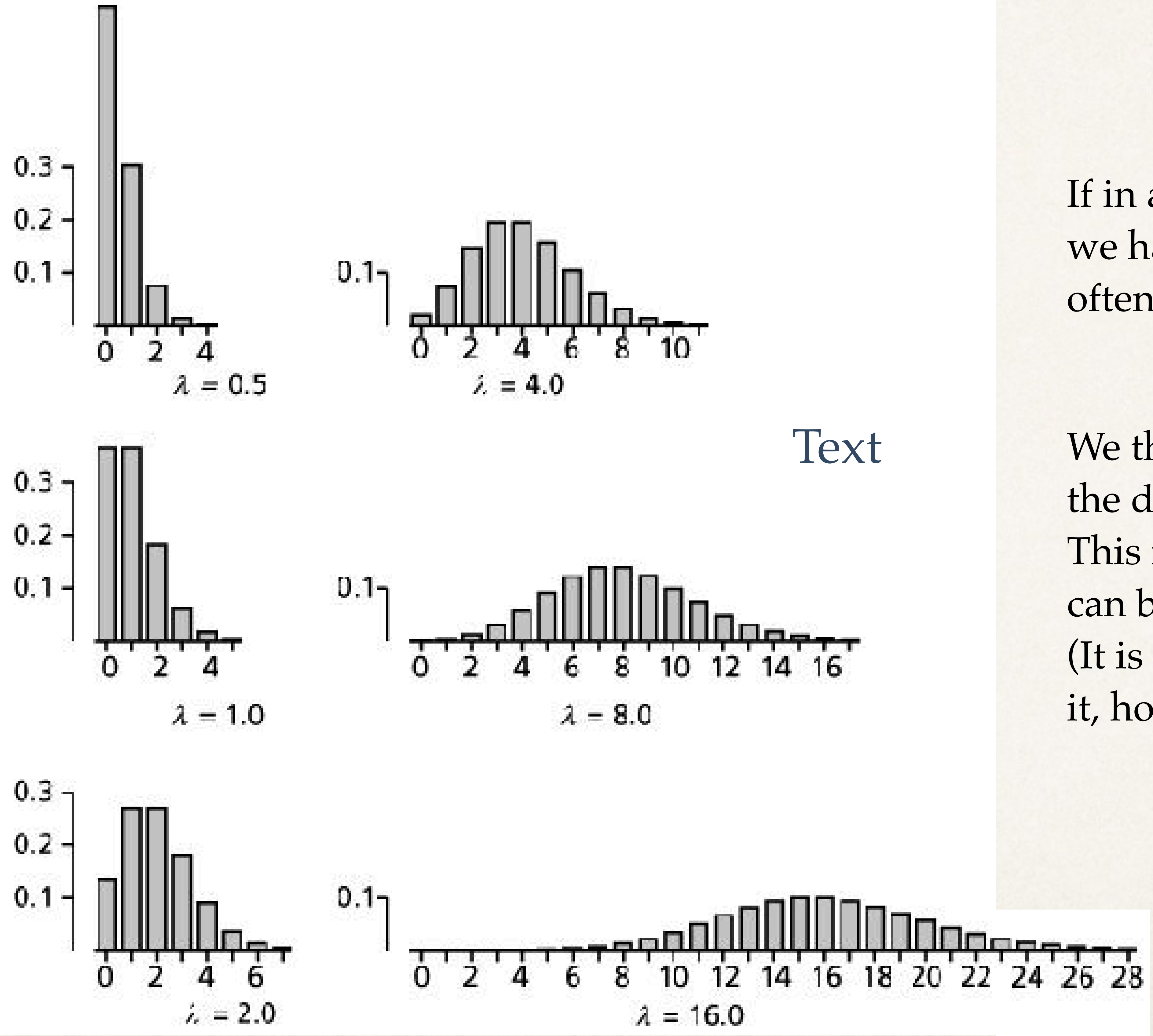


`TMath::Prob(δ²,1)`

α	δ	α	δ
0.3173	1σ	0.2	1.28σ
4.55×10^{-2}	2σ	0.1	1.64σ
2.7×10^{-3}	3σ	0.05	1.96σ
6.3×10^{-5}	4σ	0.01	2.58σ
5.7×10^{-7}	5σ	0.001	3.29σ
2.0×10^{-9}	6σ	10^{-4}	3.89σ

Table 36.1: Area of the tails α outside $\pm\delta$ from the mean of a Gaussian distribution.

Poisson for large μ is approximately Gaussian of width $\sigma = \sqrt{\mu}$



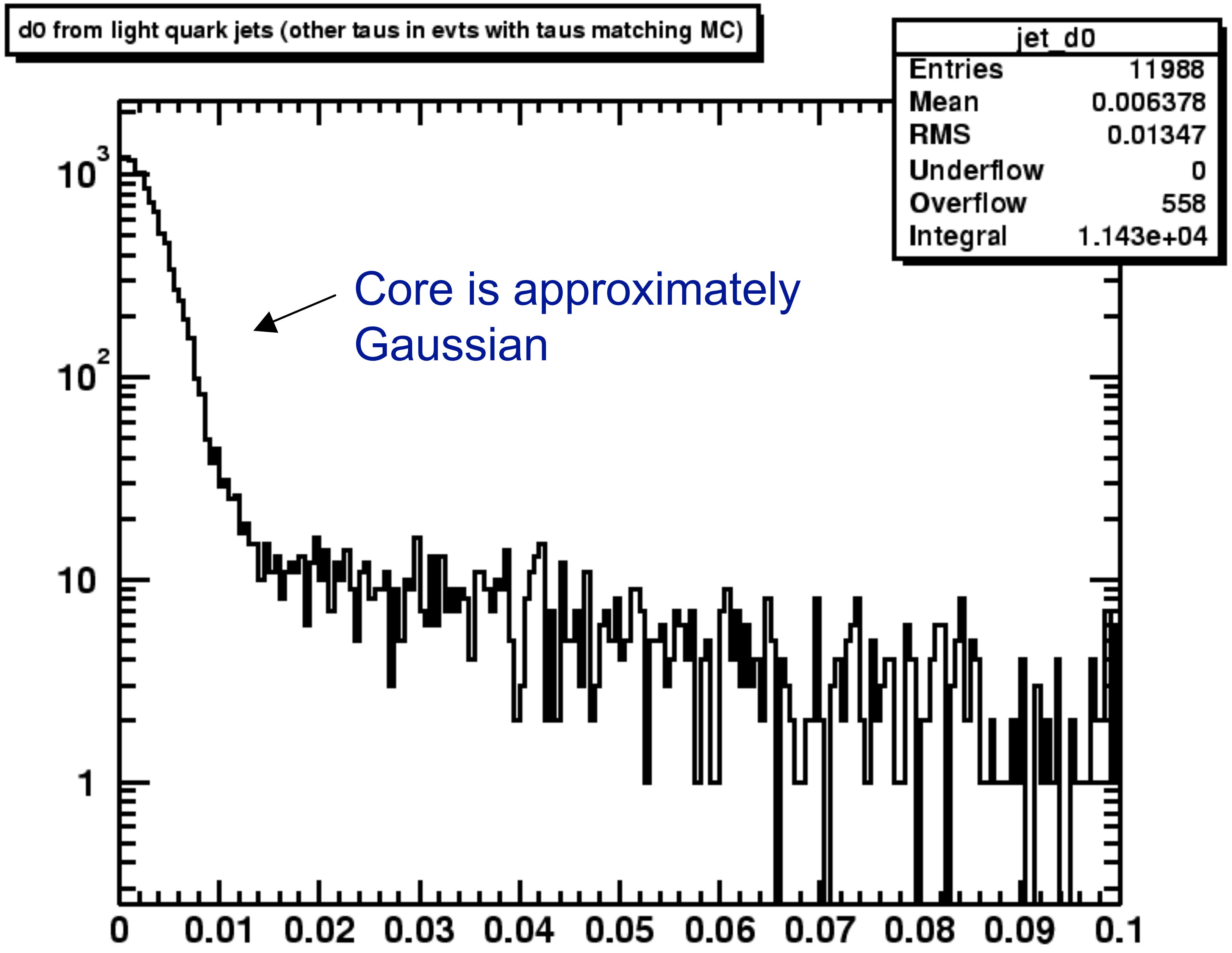
Text

If in a counting experiment all we have is a measurement n, we often use this to estimate μ.

We then draw $\sqrt{n}$ error bars on the data.
This is just a convention, and can be misleading.
(It is still recommended you do it, however.)

Not all Distributions are Gaussian

Track impact parameter distribution for example



Core is approximately Gaussian

Multiple scattering -- core: Gaussian; rare large scatters; heavy flavor, nuclear interactions, decays (taus in this example)

"All models are false. Some models are useful."

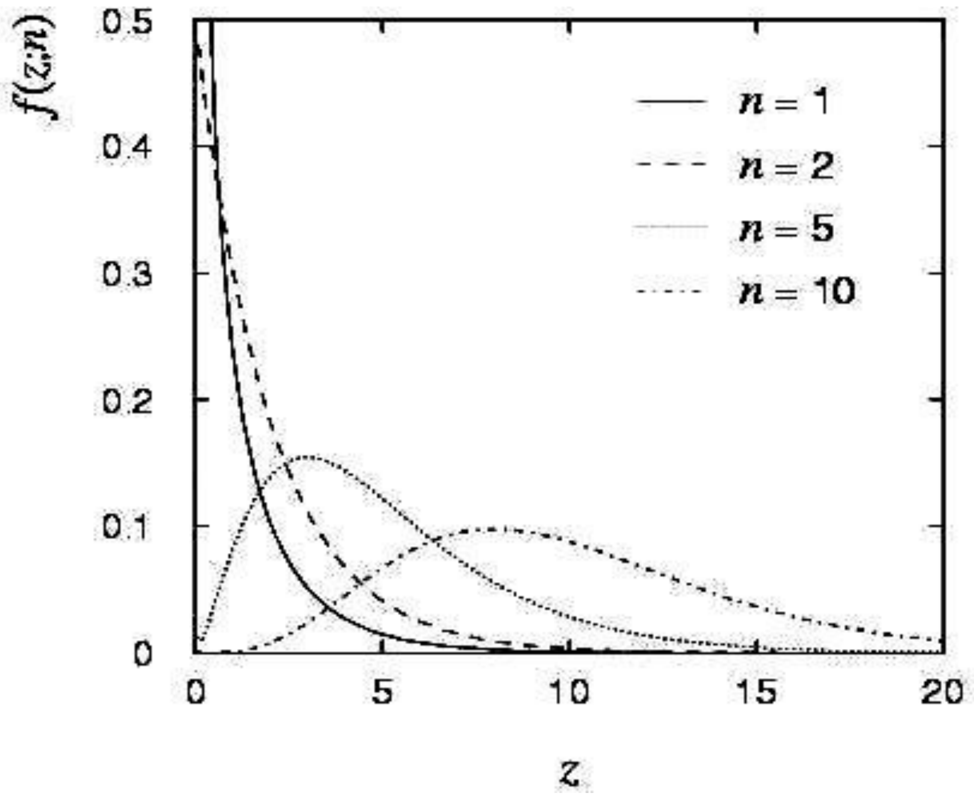
Chi-square (χ^2) distribution

The chi-square pdf for the continuous r.v. z ($z \geq 0$) is defined by

$$f(z;n) = \frac{1}{2^{n/2}\Gamma(n/2)} z^{n/2-1} e^{-z/2}$$

$n = 1, 2, ... =$ number of 'degrees of freedom' (dof)

$$E[z] = n\,, \qquad V[z] = 2n\,.$$



For independent Gaussian x_i, $i = 1, ..., n$, means μ_i, variances σ_i^2,

$$z = \sum_{i=1}^{n} \frac{(x_i - \mu_i)^2}{\sigma_i^2}$$ follows χ^2 pdf with n dof.

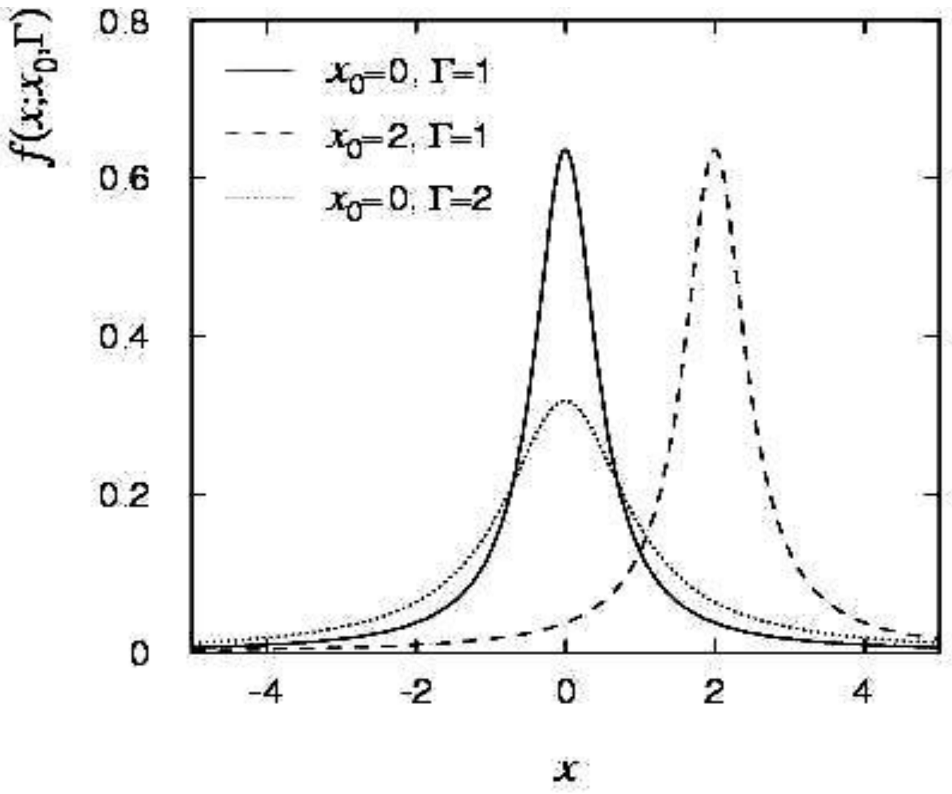
Example: goodness-of-fit test variable especially in conjunction with method of least squares.

Cauchy (Breit-Wigner) distribution

The Breit-Wigner pdf for the continuous r.v. x is defined by

$$f(x;\Gamma,x_0) = \frac{1}{\pi}\frac{\Gamma/2}{\Gamma^2/4+(x-x_0)^2}$$

($\Gamma = 2$, $x_0 = 0$ is the Cauchy pdf.)



$E[x]$ not well defined, $V[x] \rightarrow \infty$.

x_0 = mode (most probable value)

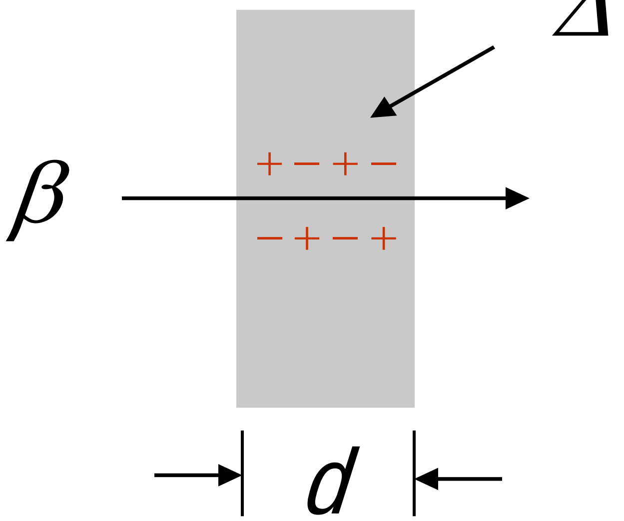
Γ = full width at half maximum

Example: mass of resonance particle, e.g. ρ, K^*, ϕ^0, ...

Γ = decay rate (inverse of mean lifetime)

Landau distribution

For a charged particle with $\beta = v/c$ traversing a layer of matter of thickness d, the energy loss Δ follows the Landau pdf:



$$f(\Delta;\beta) = \frac{1}{\xi}\phi(\lambda) ,$$

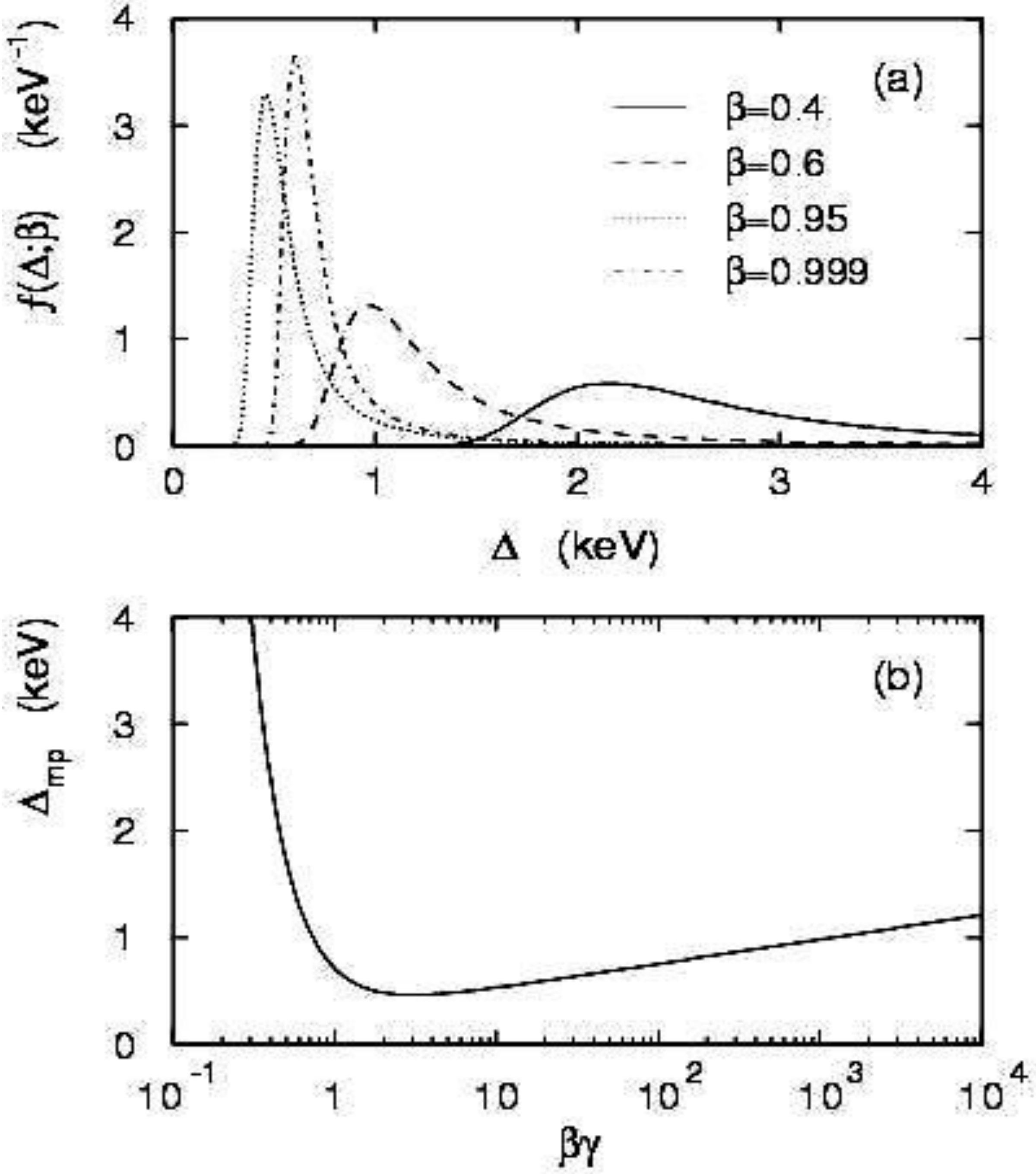
$$\phi(\lambda) = \frac{1}{\pi}\int_0^\infty \exp(-u \ln u - \lambda u) \sin \pi u \, du ,$$

$$\lambda = \frac{1}{\xi}\left[\Delta - \xi\left(\ln\frac{\xi}{\epsilon'} + 1 - \gamma_{\mathrm{E}}\right)\right] ,$$

$$\xi = \frac{2\pi N_{\mathrm{A}} e^4 z^2 \rho \sum Z}{m_{\mathrm{e}} c^2 \sum A}\frac{d}{\beta^2} , \qquad \epsilon' = \frac{I^2 \exp \beta^2}{2 m_{\mathrm{e}} c^2 \beta^2 \gamma^2} .$$

L. Landau, J. Phys. USSR 8 (1944) 201; see also
W. Allison and J. Cobb, Ann. Rev. Nucl. Part. Sci. 30 (1980) 253.

Landau distribution (2)



Long 'Landau tail'

→ all moments ∞

Mode (most probable value) sensitive to β,

→ particle i.d.

some theorems, laws…

the Law of Large Numbers

- Suppose you have a sequence of indep't random variables x_i
  - with the same mean μ
  - and variances σ_i^2
  - but otherwise distributed "however"
  - the variances are not too large

$$\lim_{N\to\infty} (1/N^2) \sum_{i=1}^{N} \sigma_i^2 = 0 \qquad (1)$$

Then the average $\bar{x}_N = (1/N)\sum_i x_i$ converges to the true mean μ

- (Note) What if the condition (1) is finite but non-zero?

$\Rightarrow$ the convergence is "almost certain" (*i.e.* the failures have measure zero)

In short, if you try many times, eventually you get the true mean!

the Central Limit Theorem

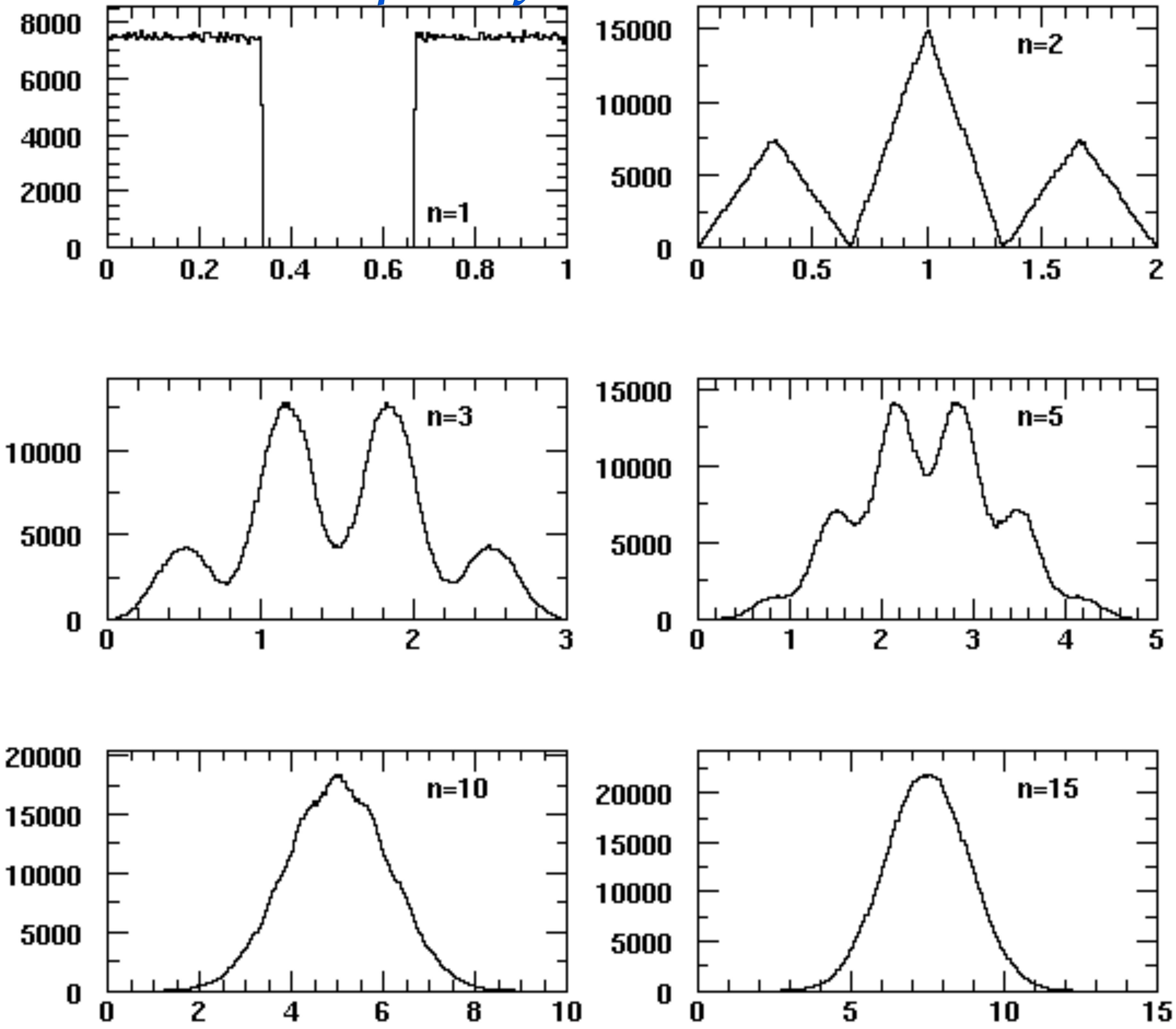
- Suppose you have a sequence of indep't random variables x_i
  - with means μ_i and variances σ_i^2
  - but otherwise distributed "however"
  - and under certain conditions on the variances

The sum $S = \sum_i x_i$ converges to a Gaussian

$$\lim_{N \to \infty} \frac{S - \sum \mu_i}{\sqrt{\sum \sigma_i^2}} \to \mathcal{N}(0, 1) \tag{2}$$

- (Note) important not to confuse LLN with CLT
  - **LLN**: with enough samples, the average → the true mean
  - **CLT**: if you put enough random numbers into your processor, the distribution of their average $\to \mathcal{N}(0, 1)$

an example of the CLT at work



the Neyman-Pearson Lemma

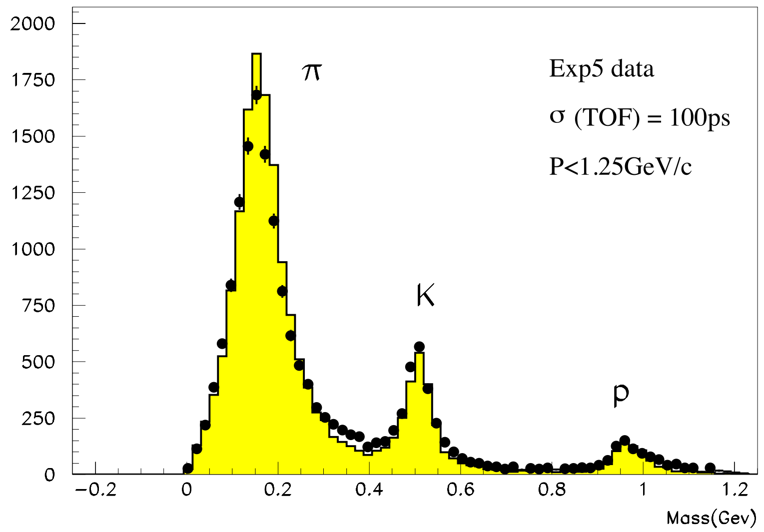
- *We will explain it later when we discuss the "critical region" ...*

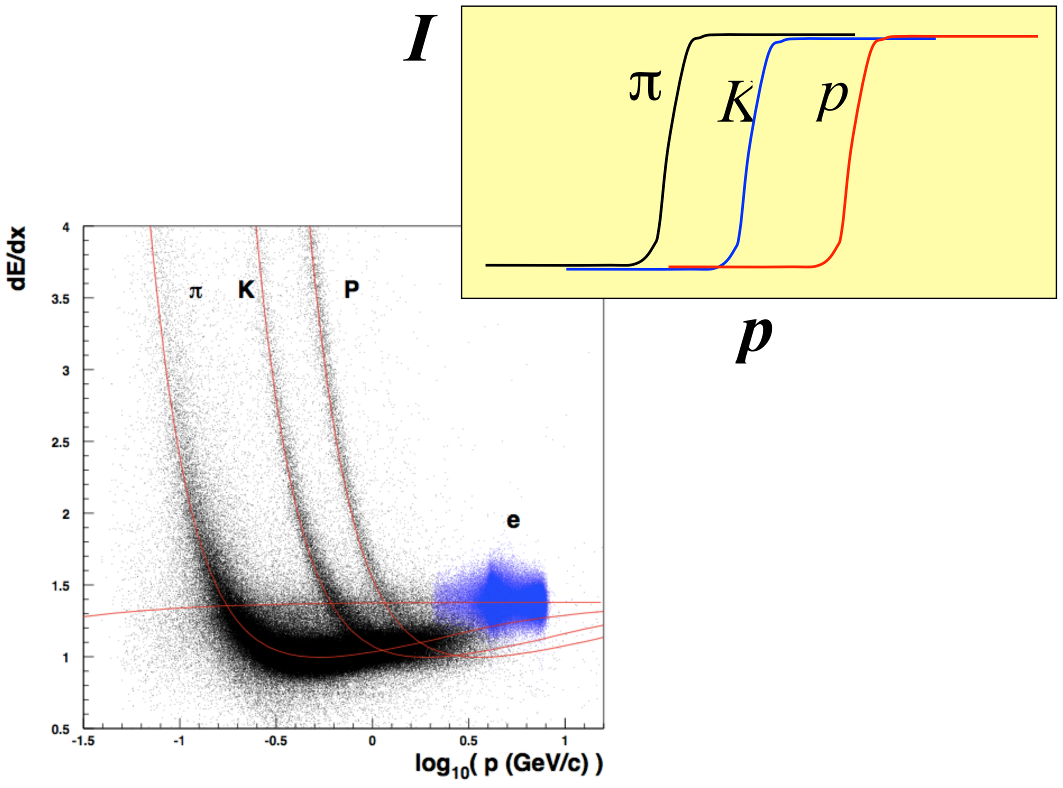
Particle identification with the `atc_pid` class is based on the likelihood of the detector response being due to an hypothesized signal particle species, compared to the likelihood for an assumed background particle species. This is expressed as a likelihood ratio

$$Prob(i:j) = \frac{P_i}{P_i + P_j}$$

$$P_i = P_i^{dE/dx} \times P_i^{TOF} \times P_i^{ACC}$$

where P_i is the particle-ID likelihood calculated for the signal particle species and P_j for the background particle species; i and j can be any of five particle species, e, μ, π, K and p. Clearly $Prob(i:j)$ is distributed on the interval $[0,1]$, and we usually think of it as





the Wilk's theorem

- *We will explain it later when we discuss the "likelihood ratio" …*

the Neyman-Pearson Lemma

- ***We will explain it later when we discuss the “critical region” …***

Hypothesis Testing

Remember?

Two approaches

Consider a set S with subsets A, B, ...

For all $A \subset S, P(A) \geq 0$

$$P(S) = 1$$

If $A \cap B = \emptyset, P(A \cup B) = P(A) + P(B)$

- **Relative frequency**

A, B, ... are outcomes of a repeatable experiment

Frequentist

$$P(A) = \lim_{n \to \infty} \frac{\text{times outcome is } A}{n}$$

- **Subjective probability**

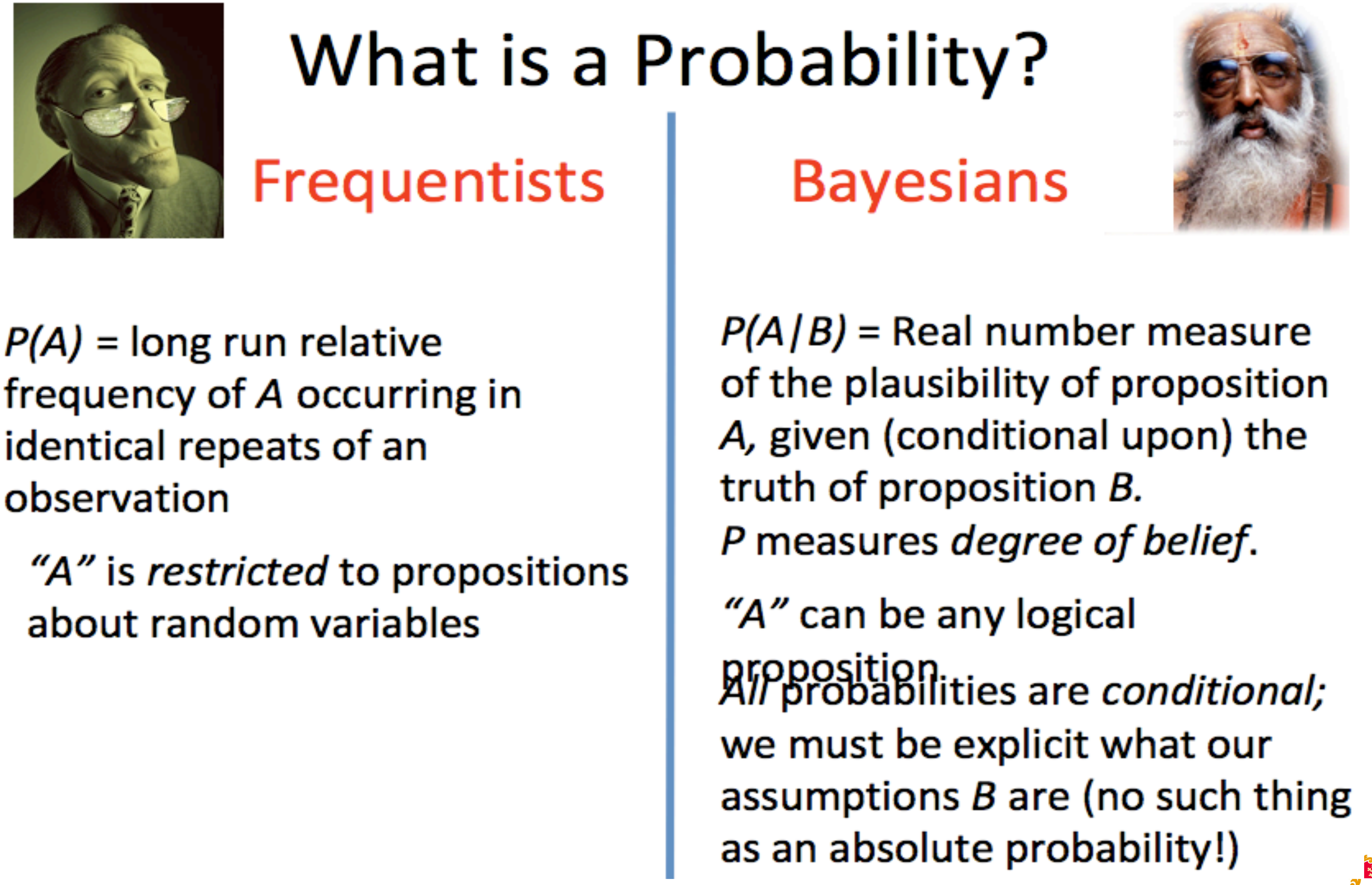
A, B, ... are hypotheses (statements that are true or false)

Bayesian

$$P(A) = \text{degree of belief that } A \text{ is true}$$

Frequentist approach is, in general, easy to understand, but some HEP phenomena are best expressed by subjective prob., e.g. systematic uncertainties, prob(Higgs boson exists), ...

Bayesian and Frequentist Statistics



What is a Probability?

Frequentists

P(A) = long run relative frequency of *A* occurring in identical repeats of an observation

"A" is *restricted* to propositions about random variables

Bayesians

P(A|B) = Real number measure of the plausibility of proposition *A*, given (conditional upon) the truth of proposition *B*.

P measures *degree of belief*.

"A" can be any logical proposition.

All probabilities are *conditional;* we must be explicit what our assumptions *B* are (no such thing as an absolute probability!)

Bayes' theorem

- **From the definition of conditional prob., we have**

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

- but $P(A \cap B) = P(B \cap A)$
- therefore,

$$P(A|B) = \frac{P(B|A)\ P(A)}{P(B)}$$

- First published (posthumous) by Ref. Thomas Bayes (1702-1761)

An essay towards solving a problem in the doctrine of chances,
Phil. Trans. R. Soc. 53 (1763) 370.

Bayesian probability: tossing a coin

- suppose I stand to win or lose money in a game of chance
- my companion gives me a coin to use in the game
- do I trust the coin?
- what is P(fair coin)?
- frequentist answer:
    - toss the coin n times
    - $P(\text{heads}) = \lim_{n\to\infty} n_H/n$
    - make a complicated statement about the results, which is *only indirectly* about whether the coin is fair (see Lec.2 . . .)
- but I can only test the coin with five throws:
    - I get 4H, 1T
    - do I trust the coin?
- frequentist answer based on these 5 trials: not much info
- Bayesian answer depends on your *prior belief* . . .
- assume for illustration that a bad coin has $P(\text{heads}) = 0.75$
- a proper analysis would involve integrating over priors, *etc.*

Bayesian probability: interpreting the coin tosses

Likelihoods:

$$P((\text{4H,1T}) \mid \text{fair}) = 0.1563$$
$$P((\text{4H,1T}) \mid \text{bad}) = 0.3955$$

Priors:

$$P(\text{fair} \mid \text{BG}) = 0.95$$
$$P(\text{bad} \mid \text{BG}) = 0.05$$

Posterior:

$$\begin{aligned} P(\text{fair} \mid (4H, 1T), \text{BG}) &= \frac{P((\text{4H,1T}) \mid \text{fair}) \cdot P(\text{fair} \mid \text{BG})}{\sum_i P((\text{4H,1T}) \mid i) \cdot P(i \mid \text{BG})} \\ &= \frac{0.1563 \cdot 0.95}{0.1563 \cdot 0.95 + 0.3955 \cdot 0.05} \\ &= 0.882 \end{aligned}$$

Bayesian probability: interpreting the coin tosses

Likelihoods:

$$P((\text{4H,1T}) \mid \text{fair}) = 0.1563$$
$$P((\text{4H,1T}) \mid \text{bad}) = 0.3955$$

Priors:

$$P(\text{fair} \mid \text{DG}) = 0.50$$
$$P(\text{bad} \mid \text{DG}) = 0.50$$

Posterior:

$$\begin{aligned} P(\text{fair} \mid (4H, 1T), \text{BG}) &= \frac{P((\text{4H,1T}) \mid \text{fair}) \cdot P(\text{fair} \mid \text{DG})}{\sum_i P((\text{4H,1T}) \mid i) \cdot P(i \mid \text{DG})} \\ &= \frac{0.1563 \cdot 0.50}{0.1563 \cdot 0.50 + 0.3955 \cdot 0.50} \\ &= 0.283 \end{aligned}$$

Bayesian probability: interpreting the coin tosses

Likelihoods:

$$\begin{aligned} P((\mathrm{8H,2T}) \mid \mathrm{fair}) &= 0.04395 \\ P((\mathrm{8H,2T}) \mid \mathrm{bad}) &= 0.28157 \end{aligned}$$

Priors:

$$\begin{aligned} P(\mathrm{fair} \mid \mathrm{DG}) &= 0.50 \\ P(\mathrm{bad} \mid \mathrm{DG}) &= 0.50 \end{aligned}$$

Posterior:

$$\begin{aligned} P(\mathrm{fair} \mid (8H, 2T), \mathrm{BG}) &= \frac{P((\mathrm{8H,2T}) \mid \mathrm{fair}) \cdot P(\mathrm{fair} \mid \mathrm{DG})}{\sum_i P((\mathrm{8H,2T}) \mid i) \cdot P(i \mid \mathrm{DG})} \\ &= \frac{0.04395 \cdot 0.50}{0.04395 \cdot 0.50 + 0.281565 \cdot 0.50} \\ &= 0.135 \end{aligned}$$

Bayesian probability: interpreting the coin tosses

Likelihoods:

$$
\begin{aligned}
P((\text{8H,2T}) \mid \text{fair}) &= 0.04395 \\
P((\text{8H,2T}) \mid \text{bad}) &= 0.28157
\end{aligned}
$$

Priors:

$$
\begin{aligned}
P(\text{fair} \mid \text{BG}) &= 0.95 \\
P(\text{bad} \mid \text{BG}) &= 0.05
\end{aligned}
$$

Posterior:

$$
\begin{aligned}
P(\text{fair} \mid (8H, 2T), \text{BG}) &= \frac{P((\text{8H,2T}) \mid \text{fair}) \cdot P(\text{fair} \mid \text{BG})}{\sum_i P((\text{8H,2T}) \mid i) \cdot P(i \mid \text{BG})} \\
&= \frac{0.04395 \cdot 0.95}{0.04395 \cdot 0.95 + 0.281565 \cdot 0.05} \\
&= 0.748
\end{aligned}
$$

Frequentist statistics – general philosophy

- In frequentist statistics, probabilities such as

  $P(\text{Higgs boson exists})$

  $P(0.117 < \alpha_s < 0.121)$

  are either 0 or 1, but we don't have the answer

Bayesian statistics – general philosophy

- In Bayesian statistics, interpretation of probability is extended to the **degree of belief** (*i.e.* subjective).
- suitable for **hypothesis testing** (but no golden rule for priors)

probability of the data assuming hypothesis H (the likelihood)

prior probability, i.e., before seeing the data

$$P(H|\vec{x}) = \frac{P(\vec{x}|H)\pi(H)}{\int P(\vec{x}|H)\pi(H)\,dH}$$

posterior probability, i.e., after seeing the data

normalization involves sum over all possible hypotheses

- can also provide more natural handling of non-repeatable things:
  e.g. systematic uncertainties, P(Higgs boson exists)

Hypothesis testing

- A hypothesis H specifies the probability for the data (*shown symbolically as $\vec{x}$ here*), often expressed as a function $f(\vec{x}|H)$
- The measured data $\vec{x}$ could be anything:
  - observation of a single particle, a single event, or an entire experiment
  - uni-/multi-variate, continuous or discrete
- the two kinds:
  - simple (or "point") hypothesis – $f(\vec{x}|H)$ is completely specified
  - composite hypothesis – H contains unspecified parameter(s)
- The probability for $\vec{x}$ given H is also called the **likelihood** of the hypothesis, written as $L(\vec{x}|H)$

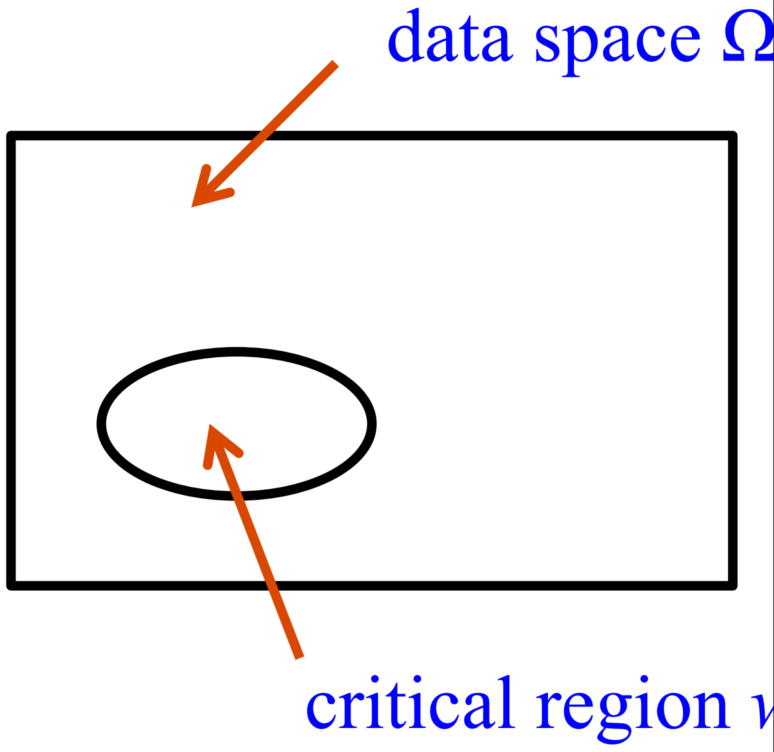
Hypothesis test

- Consider e.g. a simple hypothesis H_0 and an alternative H_1
- A (frequentist) test of H_0:

  Specify a critical region w of the data space Ω such that, assuming H_0 is correct, there is no more than some (small) probability α to observe data in w

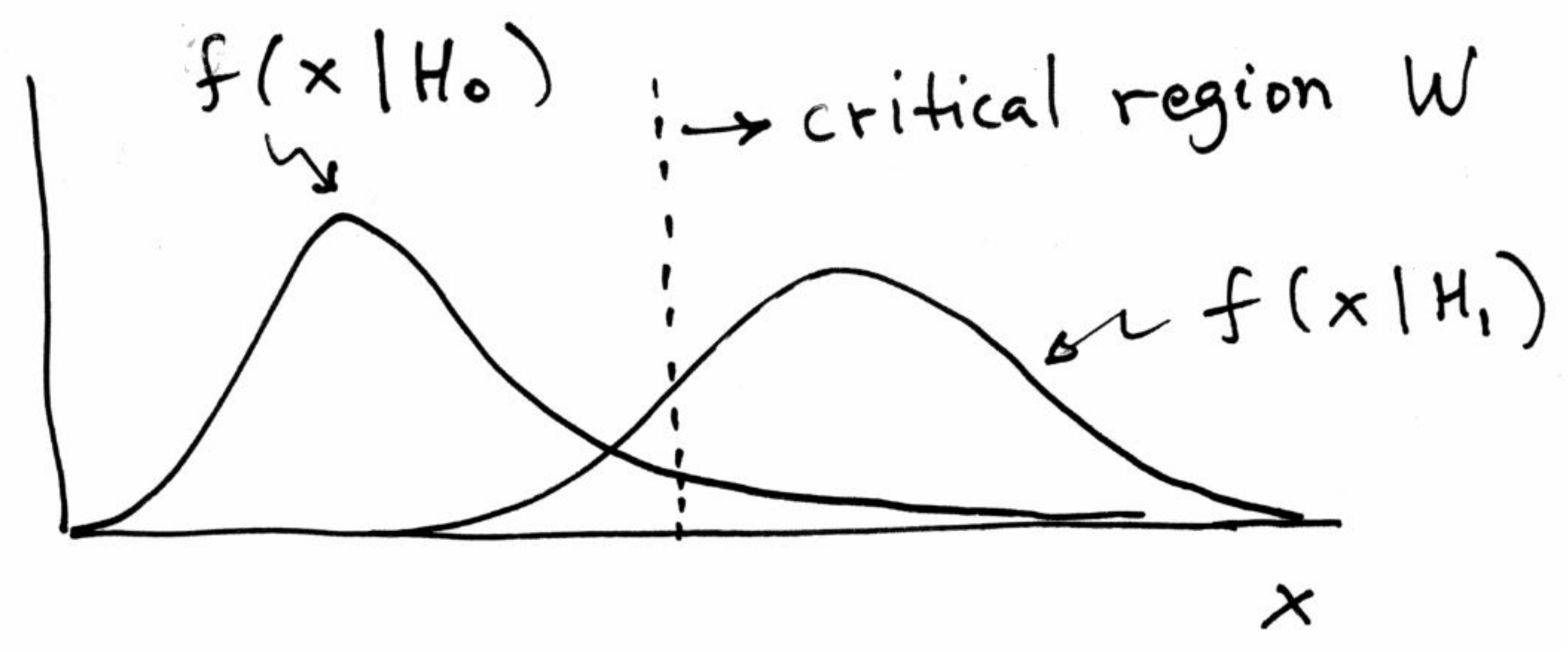
$$P(\vec{x} \in w | H_0) \leq \alpha$$

- α: "size" or "significance level" of the test
- If $\vec{x}$ is observed within w, we reject H_0 with a confidence level $1 - \alpha$



Hypothesis test

- In general, $\exists$ an ∞ number of possible critical regions that give the same significance level α
- Usually, we place the critical region where there is a low probability α for $\vec{x} \in w$ if H_0 is true, but high if the alternative (H_1) is true



Test statistic

- The boundary surface of the critical region for an n-dim. data space can be defined by an equation of the form:

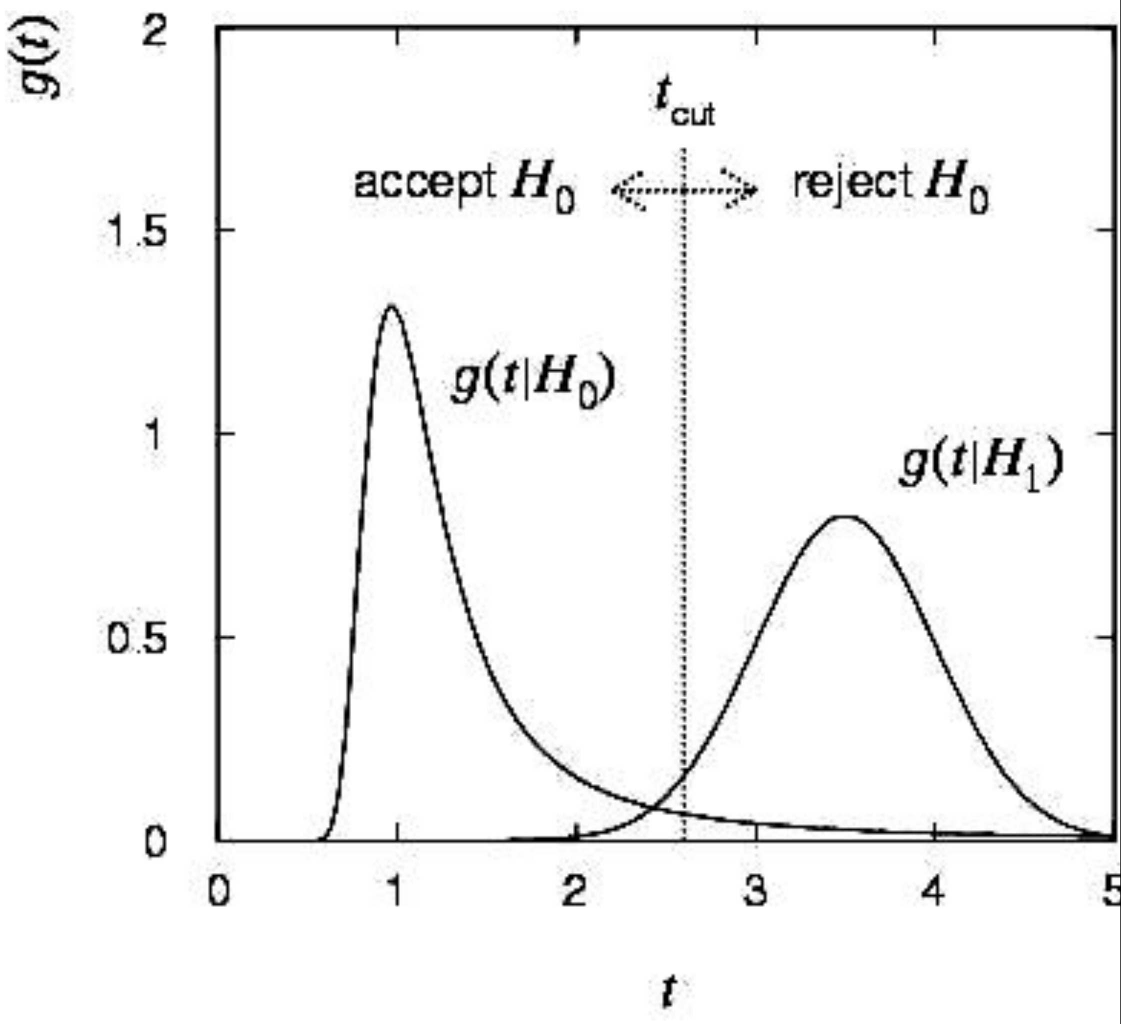
$$t(x_1, \cdots, x_n) = t_c$$

  where $t(x_1, \cdots, x_n)$ is a scalar **test statistic**.

- For the test statistic t, we can work out the PDFs $g(t|H_0)$, $g(t|H_1)$, etc.

- Decision boundary is now given by a signle 'cut' on t, thus defining the critical region

  $\Rightarrow$ for an n-dim. data space, the problem is reduced to a 1-dim. problem



Type-I, Type-II errors

- Rejecting H_0 when it is true is called the **Type-I error**

  (Q) Given the significance α of the test, what is the maximum probability of Type-I error?

- We might also accept H_0 when it is indeed false, and an alternative H_1 is true. This is called the **Type-II error**

  The probability β of Type-II error:

$$P(\vec{x} \in \Omega - w|H_1) = \beta$$

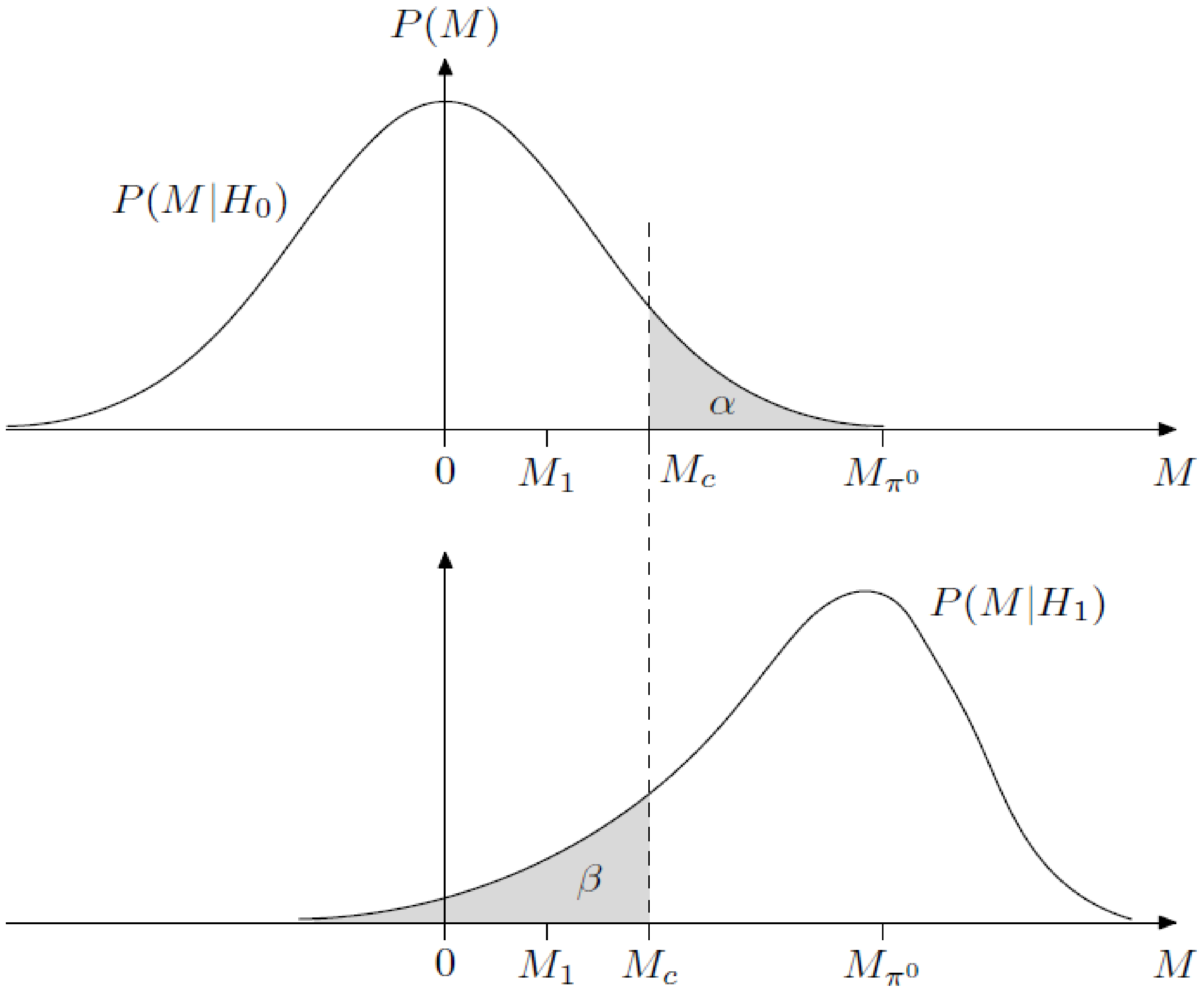
  $1 - \beta$ is called the **power** of the test with respect to H_1

Hypothesis Testing (“fixed level testing”)

Suppose we have an explicit alternative hypothesis H_1 (another statement about the model) which we may adopt if we have reason to reject H_0.

Two possible errors (Neyman-Pearson hypothesis testing):

	H_0 chosen	H_1 chosen
H_0 true	Correct decision, Prob = 1-α	Type I error, Prob = α
H_1 true	Type II error, Prob = β	Correct decision, Prob = 1-β



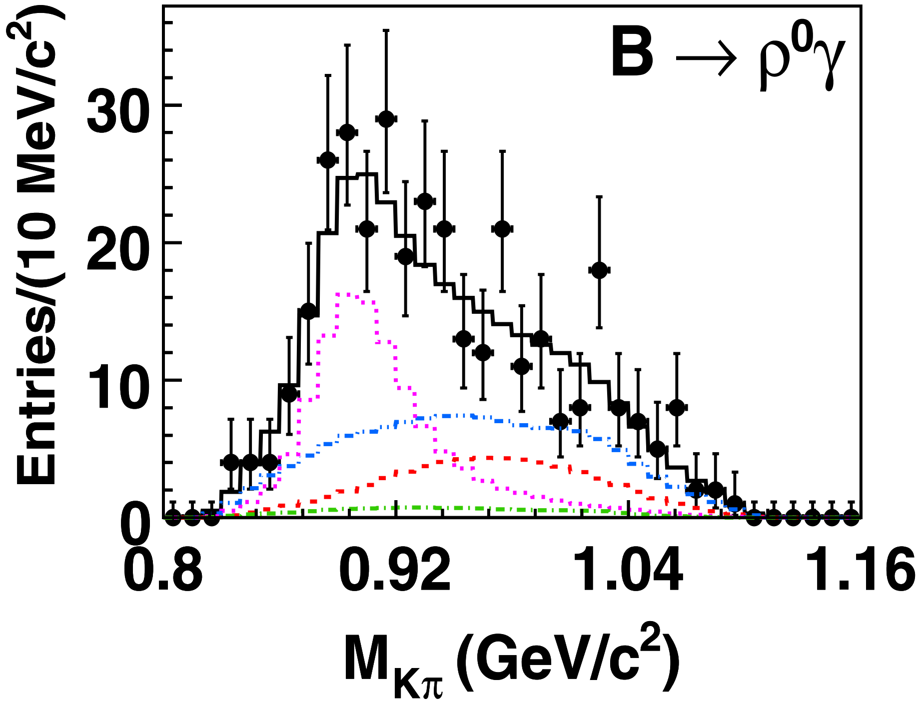
Optimal decision: minimize β for given α

- The size of the test is $\Pr_0(Y \in R_\alpha) = \alpha$.
- The power of the test is $\Pr_1(Y \in R_\alpha) = 1-\beta$.

from an FAPPS09 Lecture by S. T'Jampens

exercise on Type-I, II errors

Since $B \to K^*\gamma$ has much higher branching fraction than $B \to \rho\gamma$, the former can be a serious background to the latter. It is crucial to understand the "efficiency" and "fake rate" of K/π identification system of your experiment in this study. The figure below shows the $M_{K\pi}$ invarianbt mass distribution, where one of the pion mass (in $\rho^0 \to \pi^+\pi^-$ decay) is replaced by the Kaon mass, for the $B^0 \to \rho^0\gamma$ signal candidates (Belle, PRL 2008).



Express the following observables in Type-I & Type-II errors.

- $f_{\pi+\to K+}$ = probability of misidentifying a π^+ as a K^+
- $f_{K+\to\pi+}$ = probability of misidentifying a K^+ as a π^+
- ϵ_{K+} = prob. of identifying a K^+ correctly as a K^+
- $\epsilon_{\pi+}$ = prob. of identifying a π^+ correctly as a π^+

Probability $P(H|\vec{x})$

- In the frequentist approach, we do not, in general, assign probability of a hypothesis itself.

  Rather, we compute the probability to accept/reject a hypothesis assuming that it (or some alternative) is true.

- In Bayesian, on the other hand, probability of any given hypothesis (*degree of belief*) could be obtained by using the Bayes' theorem:

$$P(H|\vec{x}) = \frac{P(\vec{x}|H)\pi(H)}{\int P(\vec{x}|H')\pi(H')dH'}$$

  which depends on the prior probability $\pi(H)$

How to choose an *optimal* test statistic

- Use **Neyman-Pearson lemma**

  For a test of size α of the simple hypothesis H_0,
  to obtain the highest power w.r.t. the simple alternative H_1,
  choose the critical region w such that the likelihoot ratio satisfies

$$\frac{P(\vec{x}|H_1)}{P(\vec{x}|H_0)} \geq k$$

  everywhere in w and is $< k$ elsewhere,
  where k is a constant chosen for each pre-determined size α.

- Equivalently, the optimal scalar test statistic is

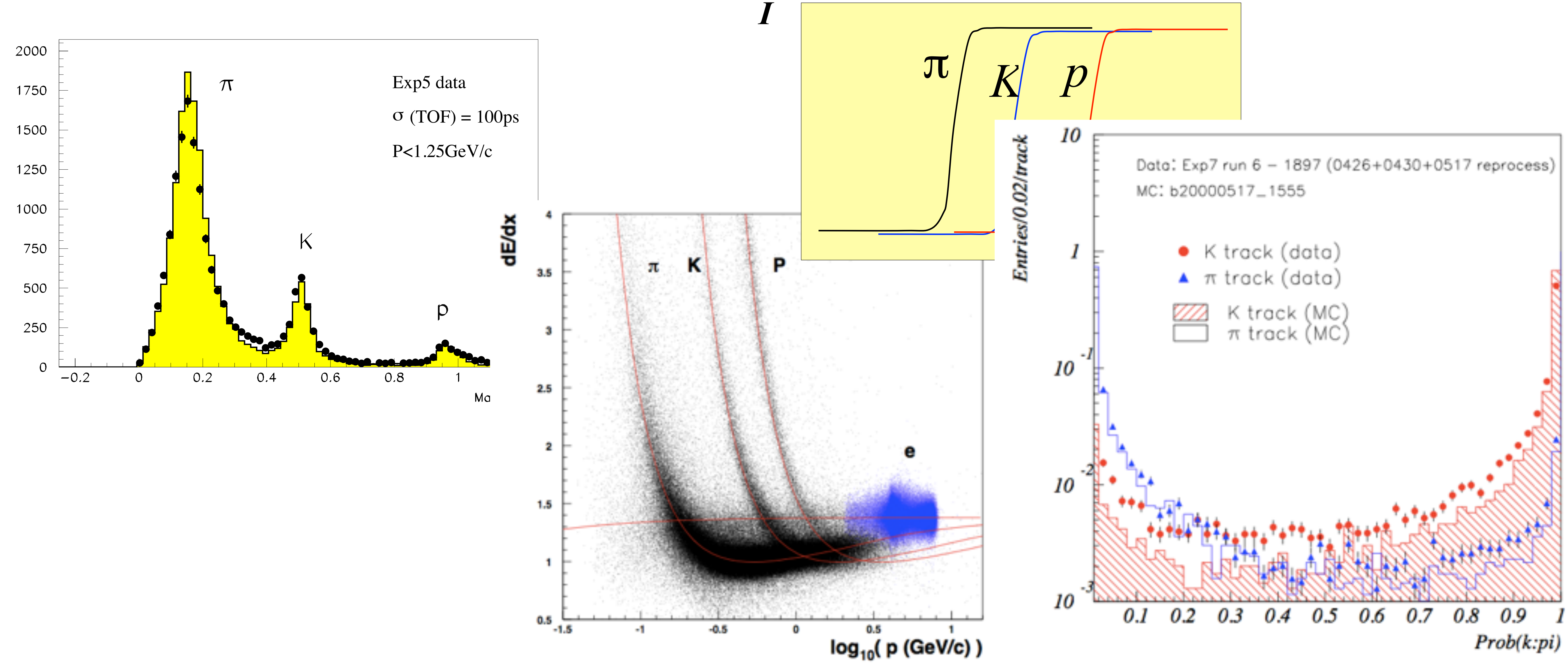
$$t(\vec{x}) = P(\vec{x}|H_1)/P(\vec{x}|H_0)$$

(Note) Any monotonic function of this leads to the *same test*.

Particle identification with the `atc_pid` class is based on the likelihood of the detector response being due to an hypothesized signal particle species, compared to the likelihood for an assumed background particle species. This is expressed as a likelihood ratio

$$Prob(i:j) = \frac{P_i}{P_i + P_j} \qquad P_i = P_i^{dE/dx} \times P_i^{TOF} \times P_i^{ACC}$$

where P_i is the particle-ID likelihood calculated for the signal particle species and P_j for the background particle species; i and j can be any of five particle species, e, μ, π, K and p. Clearly $Prob(i:j)$ is distributed on the interval $[0, 1]$, and we usually think of it as



the *p*-value

- With *p*-value, we express the level of agreement b/w data and H

  $p =$ probabilty, under assumption of H, to observe data with equal or lesser compatibility with H, in comparison to the data we obtained

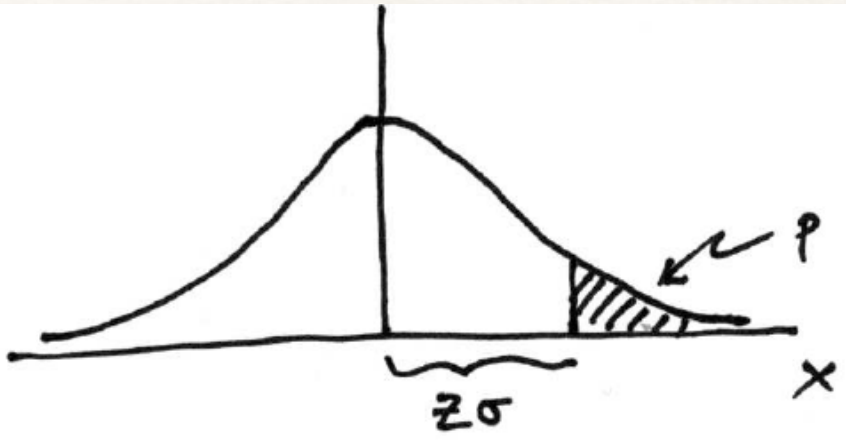
  $\neq$ *the probability that H is true* ⚠
- In frequentist statistics, we don't talk about $P(H)$.

  In Bayesian, however, we determine $P(H|\vec{x})$ using the Bayes' theorem

  $\Leftarrow$ depending on the prior probabilty $\pi(H)$
- For now, we stick with the frequentist interpretation of the *p*-value

Significance from the *p*-value

- **Often we quote the significance Z, for a given p-value**
  - Z = the number of standard dev. that a Gaussian random variable would fluctuate in one direction to give the same *p*-value



$$p = \int_Z^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 1 - \Phi(Z)$$

`1 - TMath::Freq`

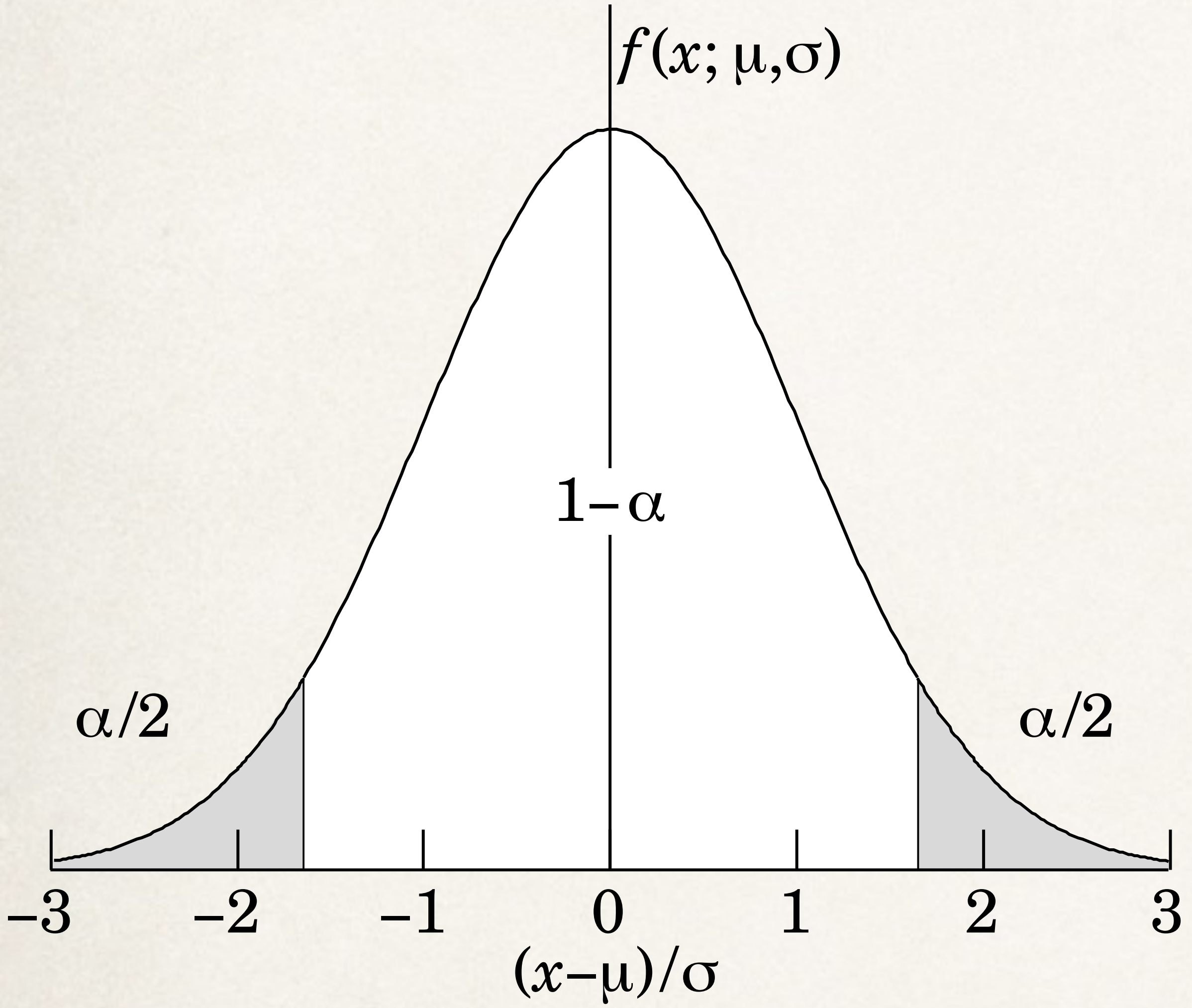
$$Z = \Phi^{-1}(1-p)$$

`TMath::NormQuantile`

(Ex) $Z = 5$ (a “5-sigma effect”) ⇔ $p = 2.9 \times 10^{-7}$

Remember?

Gaussian (Normal) distribution



`TMath::Prob(δ²,1)`

α	δ	α	δ
0.3173	1σ	0.2	1.28σ
4.55×10^{-2}	2σ	0.1	1.64σ
2.7×10^{-3}	3σ	0.05	1.96σ
6.3×10^{-5}	4σ	0.01	2.58σ
5.7×10^{-7}	5σ	0.001	3.29σ
2.0×10^{-9}	6σ	10^{-4}	3.89σ

Table 36.1: Area of the tails α outside $\pm\delta$ from the mean of a Gaussian distribution.

(Ex) $Z = 5$ (a "5-sigma effect") ⇔ $p = 2.9 \times 10^{-7}$

p-value example: testing whether a coin is 'fair'

Probability to observe n heads in N coin tosses is binomial:

$$P(n; p, N) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

Hypothesis H: the coin is fair ($p = 0.5$).

Suppose we toss the coin $N = 20$ times and get $n = 17$ heads.

Region of data space with equal or lesser compatibility with H relative to $n = 17$ is: $n = 17, 18, 19, 20, 0, 1, 2, 3$. Adding up the probabilities for these values gives:

$$P(n = 0, 1, 2, 3, 17, 18, 19, \text{ or } 20) = 0.0026\ .$$

i.e. $p = 0.0026$ is the probability of obtaining such a bizarre result (or more so) 'by chance', under the assumption of H.

The significance of an observed signal

Suppose we observe n events; these can consist of:

n_b events from known processes (background)
n_s events from a new process (signal)

If n_s, n_b are Poisson r.v.s with means s, b, then $n = n_\mathrm{s} + n_\mathrm{b}$ is also Poisson, mean = $s + b$:

$$P(n; s, b) = \frac{(s+b)^n}{n!} e^{-(s+b)}$$

Suppose $b = 0.5$, and we observe $n_\mathrm{obs} = 5$. Should we claim evidence for a new discovery?

Give p-value for hypothesis $s = 0$:

$$\begin{aligned} p\text{-value} &= P(n \geq 5; b = 0.5, s = 0) \\ &= 1.7 \times 10^{-4} \;\; \neq P(s = 0)! \end{aligned}$$

The significance of an observed signal

Suppose we observe n events; these can consist of:

n_b events from known processes (background)
n_s events from a new process (signal)

If n_s, n_b are Poisson r.v.s with means s, b, then $n = n_\mathrm{s} + n_\mathrm{b}$ is also Poisson, mean $= s + b$:

$$P(n; s, b) = \frac{(s+b)^n}{n!} e^{-(s+b)}$$

Suppose $b = 0.5$, and we observe $n_\mathrm{obs} = 5$. Should we claim evidence for a new discovery?

Give p-value for hypothesis $s = 0$:

$$\begin{aligned} p\text{-value} &= P(n \geq 5; b = 0.5, s = 0) \\ &= 1.7 \times 10^{-4} \neq P(s = 0)! \end{aligned}$$

Quiz

1983 Champion Haitai Tigers starting roster

			타율	출루율	장타율	홈런	타점	도루
1	김일권	CF	.275	.345	.364	6	26	48
2	서정환	SS	.257	.320	.339	3	34	13
3	김성한	1B	.327	.401	.448	7	40	13
4	김봉연	DH	.280	.371	.552	22	59	2
5	김종모	LF	.350	.404	.524	11	44	7
6	김준환	RF	.248	.308	.362	10	43	11
7	김무종	C	.262	.313	.453	12	60	2
8	양승호	3B	.236	.292	.309	2	11	3
9	차영화	2B	.266	.308	.323	1	23	16

1983 Korean Baseball Champion Haitai Tigers starting roster

- (observation) Six out of 9 starting hitters have family name ‘Kim’.
- (fact) According to census, ~20% of all Koreans have family name ‘Kim’.
- (Hypothesis to test) The manager of 1983 Tigers (himself a ‘Kim’) has a bias toward players with family name ‘Kim’.

Model-independent test?

- In general, we cannot find a single critical region that gives tha maximum power for all possible alternatives (no "uniformly most powerful" test)
- In HEP, we often try to construct a test of the Standard Model as H_0 (or sometimes called "background only")

  such that we have a well specified *false discovery rate* α (=prob. to reject H_0 when it is true),

  and high power w.r.t. some interesting alternative H_1, e.g. SUSY, Z', etc.
- But, there is no such thing as a *model-independent* test.

  Any statistical test will inevitably have high power w.r.t. some alternatives and less for others

Confidence interval from inversion of a test

- Suppose a model contains a parameter μ

  We want to know which values are consistent with data and which are disfavored.

- Carry out a test of size α for all values of μ.
- The values that are *not rejected* constitutes a **confidence interval** for μ at confidence level $\mathrm{CL} = 1 - \alpha$.

  The probability that the true value of μ will be rejected is not greater than α, so by construction the confidence interval will contain the true value of μ with probability $\geq 1 - \alpha$.

  - The interval depends on the choice of the test (critical region).
  - It the test is formulated in terms of a p-value, p_μ, then the confidence interval represents those values of μ for which $p_\mu > \alpha$.
  - To find the end points of the interval, set $p_\mu = \alpha$ and solve for μ.

(Ex) UL on Poisson parameter

Consider again the case of observing $n \sim \text{Poisson}(s + b)$.

Suppose $b = 4.5$, $n_{\text{obs}} = 5$. Find upper limit on s at 95% CL.

Relevant alternative is $s = 0$ (critical region at low n)

p-value of hypothesized s is $\text{P}(n \le n_{\text{obs}}; s, b)$

Upper limit s_{up} at CL = $1 - \alpha$ found from

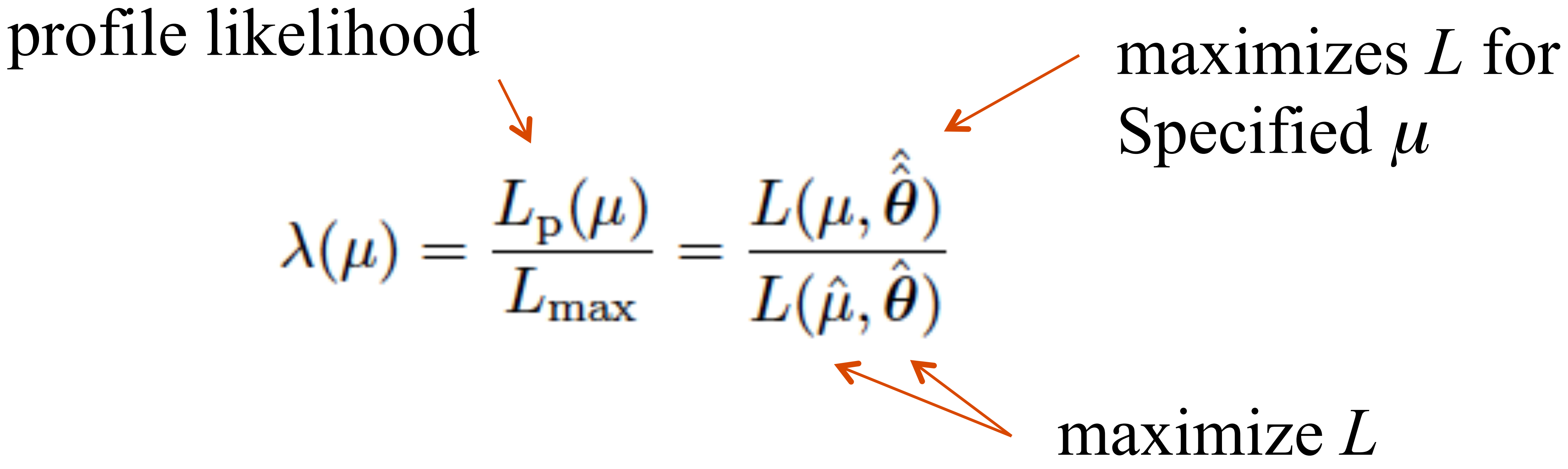
$$\alpha = P(n \le n_{\text{obs}}; s_{\text{up}}, b) = \sum_{n=0}^{n_{\text{obs}}} \frac{(s_{\text{up}} + b)^n}{n!} e^{-(s_{\text{up}}+b)}$$

$$s_{\text{up}} = \frac{1}{2} F_{\chi^2}^{-1}(1 - \alpha; 2(n_{\text{obs}} + 1)) - b$$

$$= \frac{1}{2} F_{\chi^2}^{-1}(0.95; 2(5 + 1)) - 4.5 = 6.0$$

The profile likelihood ratio

- Base significance test on the profile likelihood ratio



profile likelihood

maximizes L for Specified μ

$$\lambda(\mu) = \frac{L_{\mathrm{p}}(\mu)}{L_{\max}} = \frac{L(\mu, \hat{\hat{\theta}})}{L(\hat{\mu}, \hat{\theta})}$$

maximize L

  - the likelihood ratio of point hypotheses gives optimal test (by Neyman-Pearson lemma)
  - the statistic above is nearly optimal

- Advantage of $\lambda(\mu)$ – in large sample limit, $f(-2\ln\lambda(\mu)\,|\mu)$ approaches a χ^2 pdf for $n = 1$ (by *Wilk's theorem*)

Parameter Estimation

Basics of parameter estimation

- The parameters of a PDF are constants characterizing its shape, e.g.

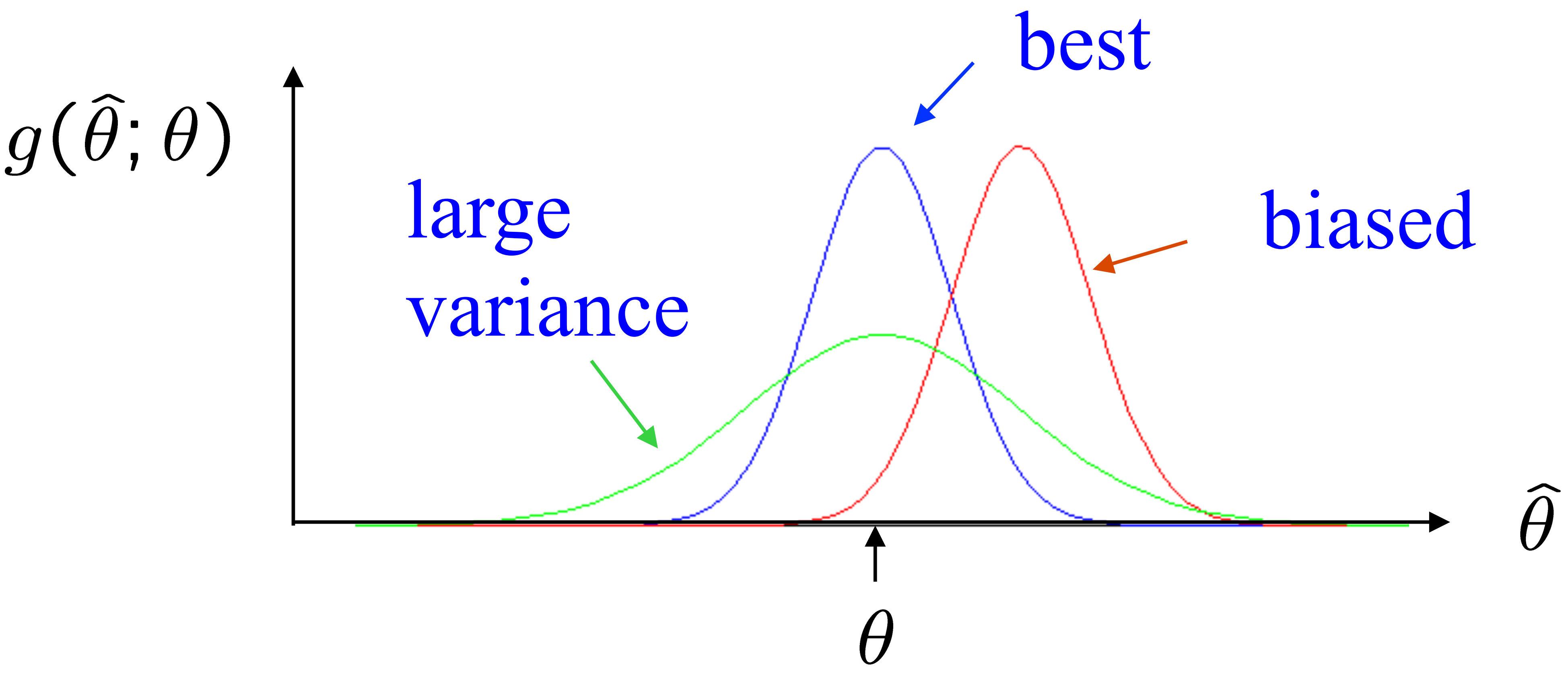
$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$$

  where θ is the parameter, while x is the random variable.

- Suppose we have a sample of observed values, $\vec{x}$.
  We want to find some function of the data to *estimate* the parameter(s): $\hat{\theta}(\vec{x})$.
  Often $\hat{\theta}$ is called an **estimator**.

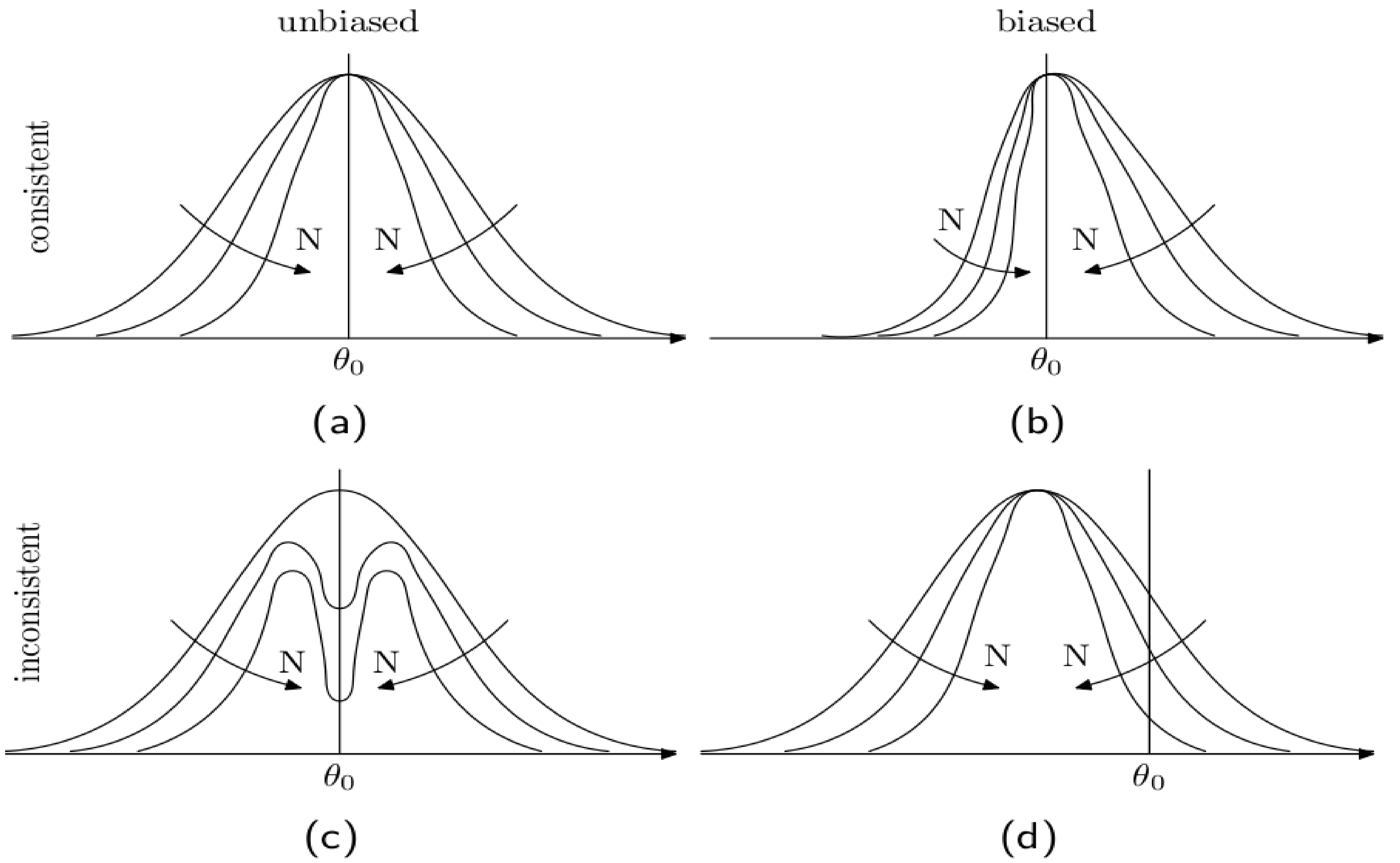
Properties of estimators

- If we were to repeat the entire measurement, the set of estimates would follow a PDF:



  - We want small (or zero) bias (⇒ syst. error): $b = E[\hat{\theta}] - \theta$
  - and we want a small variance (⇒ stat. error): $V[\hat{\theta}]$

Bias versus Consistency



N: sample size

from an FAPPS09 Lecture by S. T'Jampens

The likelihood function

- Suppose the entire result of an experiment (*set of measurements*) is a collection of numbers $\vec{x}$, and suppose the joint PDF for the data $\vec{x}$ is a function depending on a set of parameters $\vec{\theta}$: $f(\vec{x};\vec{\theta})$
- Evaluate this function with the measured data $\vec{x}$, regarding this as a function of $\vec{\theta}$ only. This is the **likelihood function**.

$$L(\vec{\theta}) = f(\vec{x};\vec{\theta}) \ (\vec{x}, \text{fixed})$$

The likelihood function for i.i.d. data

i.i.d. = *independent and identically distributed*

- Consider n independent observations of x: $x_1, \cdots, x_n$, where x follows $f(x,\theta)$. The joint PDF for the whole data sample is:

$$f(x_1, \cdots, x_n; \vec{\theta}) = \prod_{i=1}^{n} f(x_i; \vec{\theta})$$

- In this case, the likelihood function is

$$L(\vec{\theta}) = \prod_{i=1}^{n} f(x_i; \vec{\theta}) \quad (x_i \ \text{constant})$$

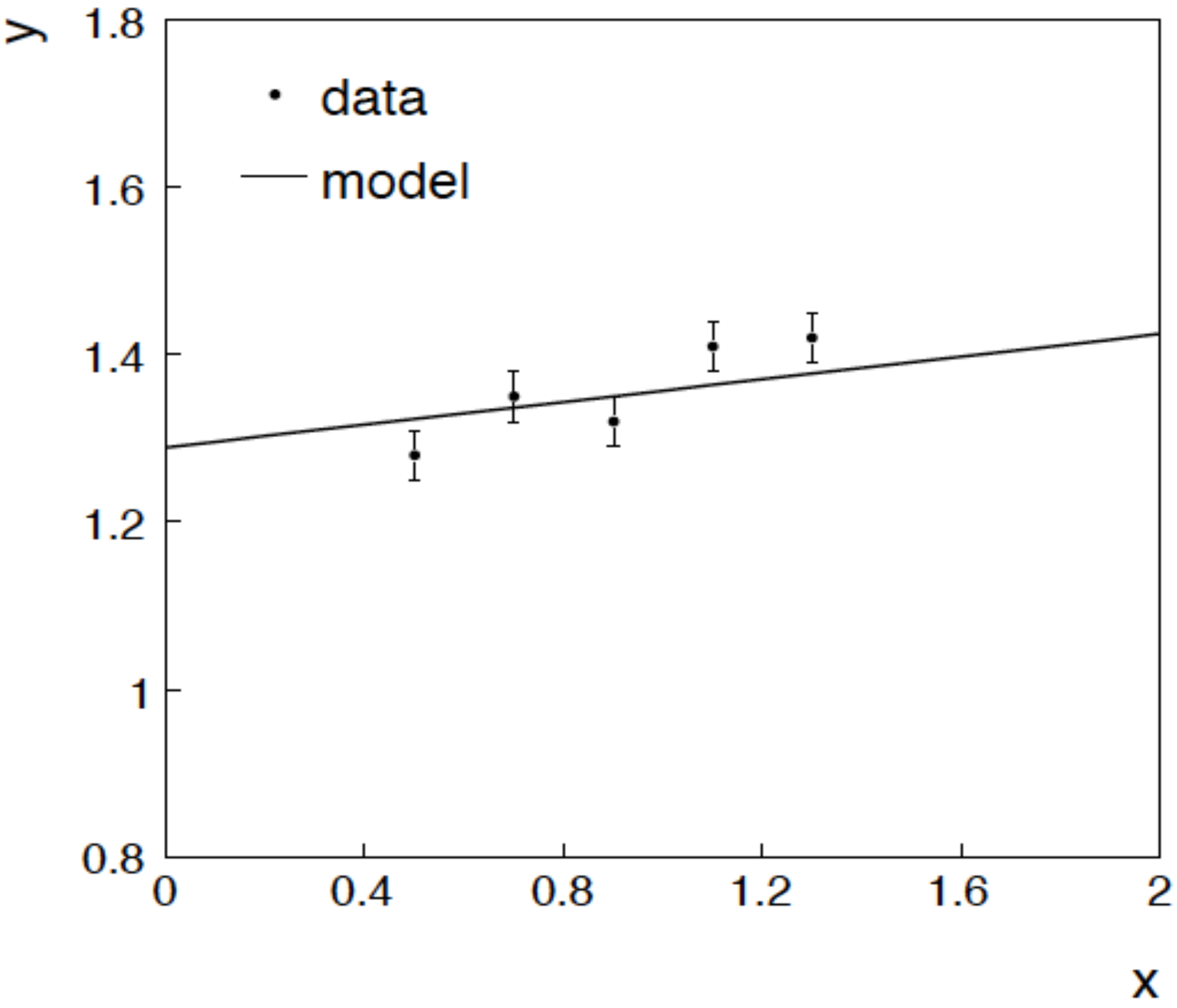
So we define the max. likelihood (ML) estimator(s) to be the parameter value(s) for which the L becomes maximum.

Example: fitting a straight line

Data: (x_i, y_i, σ_i) , $i = 1, \ldots, n$.

Model: y_i independent and all follow $y_i \sim \text{Gauss}(\mu(x_i), \sigma_i)$

$$\mu(x; \theta_0, \theta_1) = \theta_0 + \theta_1 x \ ,$$



assume x_i and σ_i known.

Goal: estimate θ_0

Here suppose we don't care about θ_1 (example of a "nuisance parameter")

Maximum likelihood fit with Gaussian data

In this example, the y_i are assumed independent, so the likelihood function is a product of Gaussians:

$$L(\theta_0, \theta_1) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{1}{2}\frac{(y_i - \mu(x_i;\theta_0,\theta_1))^2}{\sigma_i^2}\right] ,$$

Maximizing the likelihood is here equivalent to minimizing

$$\chi^2(\theta_0, \theta_1) = -2\ln L(\theta_0, \theta_1) + \text{const} = \sum_{i=1}^{n} \frac{(y_i - \mu(x_i;\theta_0,\theta_1))^2}{\sigma_i^2} .$$

i.e., for Gaussian data, ML same as Method of Least Squares (LS)

Bayesian likelihood function

- Suppose our L-function contains two parameters θ_0 and θ_1, where we have some knoweldege about the prior probability on θ_1 from previous measurements:

$$\begin{aligned} \pi(\theta_0, \theta_1) &= \pi_0(\theta_0)\pi_1(\theta_1) \\ \pi_0(\theta_0) &= \text{const.} \\ \pi_1(\theta_1) &= \frac{1}{\sqrt{2\pi}\sigma_p} e^{-(\theta_1-\theta_p)^2/2\sigma_p^2} \end{aligned}$$

- Putting this into the Bayes' theorem gives the posterior probability:

$$p(\theta_0, \theta_1|\vec{x}) \propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_i} e^{-(y_i-\mu(x_i;\theta_0,\theta_1))^2/2\sigma_i^2} \; \pi_0 \; \frac{1}{\sqrt{2\pi}\sigma_p} e^{-(\theta_1-\theta_p)^2/2\sigma_p^2}$$

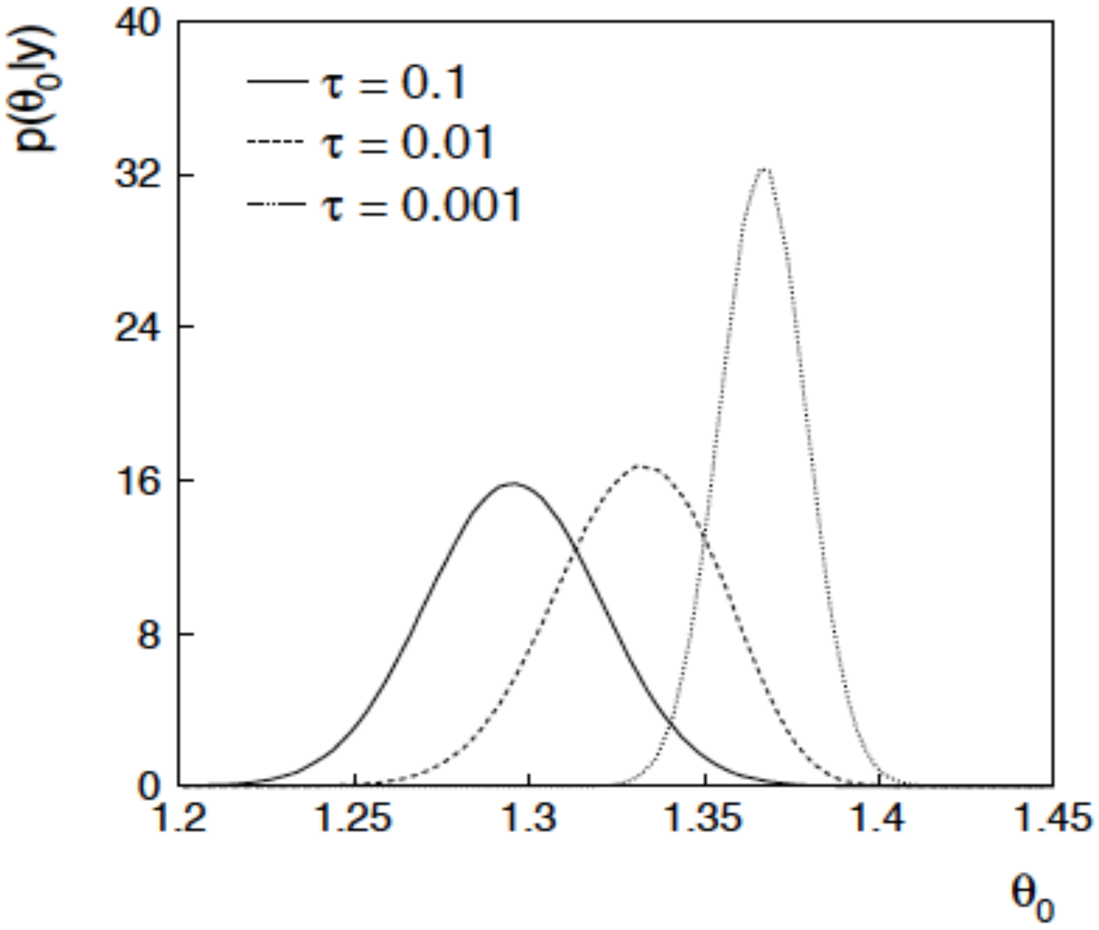
- Then, $p(\theta_0|\vec{x}) = \int p(\theta_0, \theta_1|\vec{x})\, d\theta_1$

Bayesian method with alternative priors

Suppose we don't have a previous measurement of θ_1 but rather, e.g., a theorist says it should be positive and not too much greater than 0.1 "or so", i.e., something like

$$\pi_1(\theta_1) = \frac{1}{\tau} e^{-\theta_1/\tau} , \quad \theta_1 \geq 0 , \quad \tau = 0.1 .$$

From this we obtain (numerically) the posterior pdf for θ_0:



This summarizes all knowledge about θ_0.

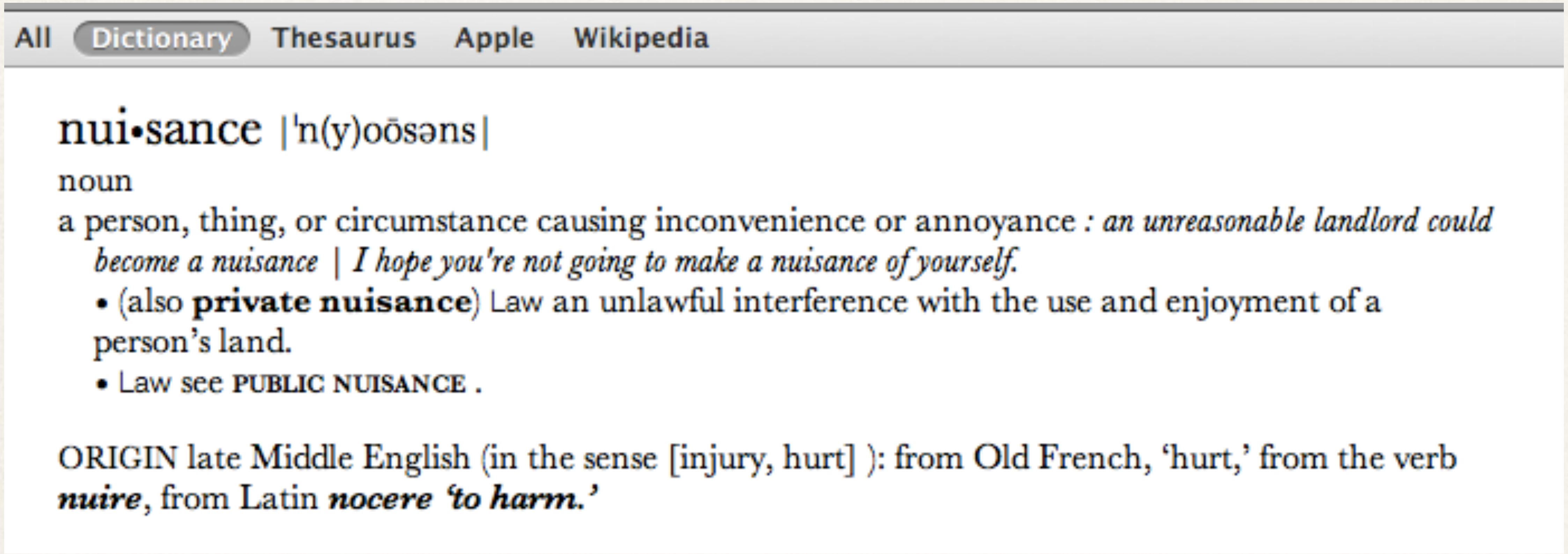
Look also at result from variety of priors.

some more sophisticated topics

- nuisance parameters & systematic uncertainties
- spurious exclusion ➔ the CL_s procedure
- look-elsewhere effect

Systematic uncertainties?

In statistics, they call it the "nuisance parameter"



nui•sance |ˈn(y)oōsəns|

noun

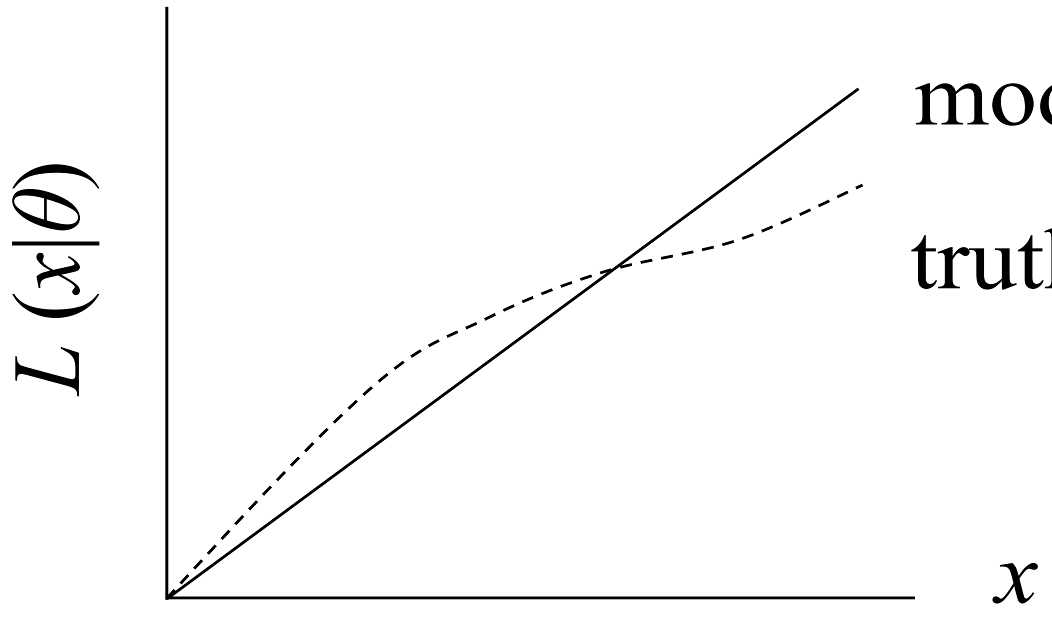
a person, thing, or circumstance causing inconvenience or annoyance : *an unreasonable landlord could become a nuisance | I hope you're not going to make a nuisance of yourself.*

- (also **private nuisance**) Law an unlawful interference with the use and enjoyment of a person's land.
- Law see **PUBLIC NUISANCE** .

ORIGIN late Middle English (in the sense [injury, hurt]): from Old French, 'hurt,' from the verb ***nuire***, from Latin ***nocere* 'to harm.'**

Nuisance parameters

In general our model of the data is not perfect:



model: $L(x|\theta) = \theta x$

truth: $L(x|\theta) = \theta x + \alpha x^2 + \beta x^3 + \cdots$

Can improve model by including additional adjustable parameters.

$$L(x|\theta) \to L(x|\theta, \nu)$$

Nuisance parameter ↔ systematic uncertainty. Some point in the parameter space of the enlarged model should be "true".

Presence of nuisance parameter decreases sensitivity of analysis to the parameter of interest (e.g., increases variance of estimate).

p-values in cases with nuisance parameters

Suppose we have a statistic q_θ that we use to test a hypothesized value of a parameter θ, such that the p-value of θ is

$$p_\theta = \int_{q_{\theta,\mathrm{obs}}}^{\infty} f(q_\theta|\theta,\nu)\, dq_\theta$$

But what values of ν to use for $f(q_\theta|\theta, \nu)$?

Fundamentally we want to reject θ only if $p_\theta < \alpha$ for all ν.

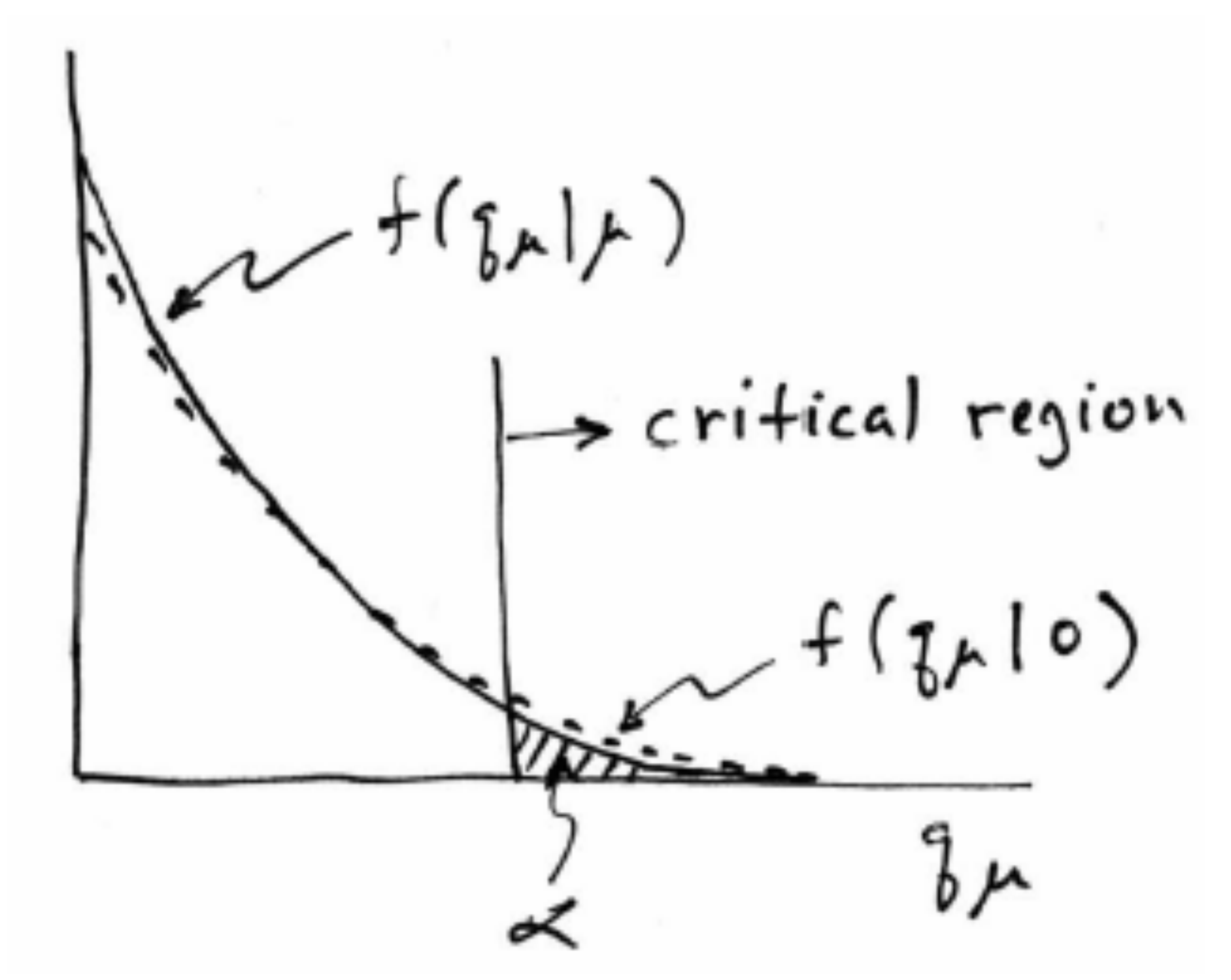
→ "exact" confidence interval

Recall that for statistics based on the profile likelihood ratio, the distribution $f(q_\theta|\theta, \nu)$ becomes independent of the nuisance parameters in the large-sample limit.

But in general for finite data samples this is not true; one may be unable to reject some θ values if all values of ν must be considered, even those strongly disfavoured by the data (resulting interval for θ "overcovers").

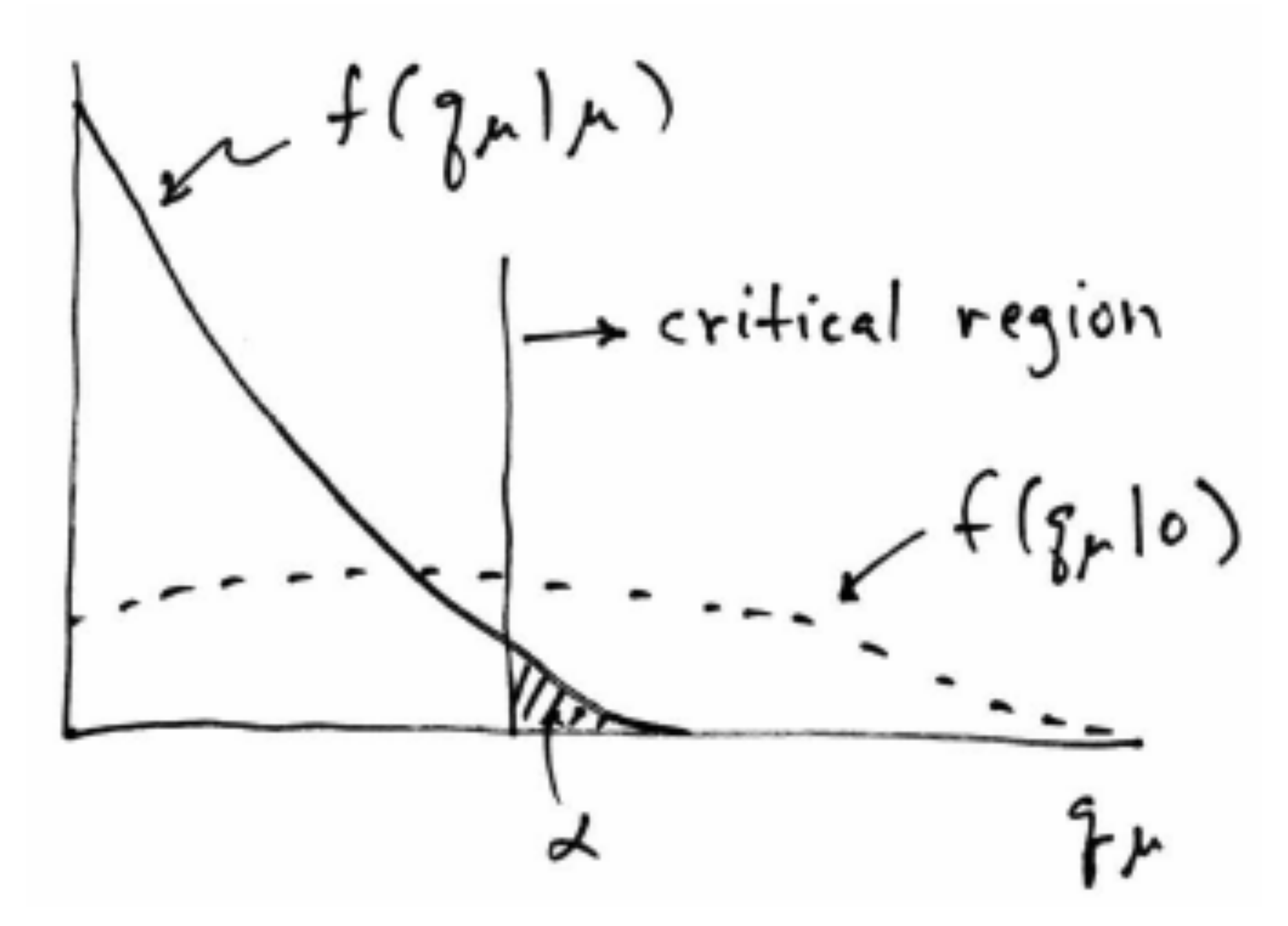
low sensitivity & spurious exclusion

- **Sometimes, the effect of a given hypothesized μ is very small relative to the null (μ =0) prediction**
  - This means that the distributions $f(q_\mu | \mu)$ and $f(q_\mu | 0)$ will be almost the same.



low sensitivity & spurious exclusion

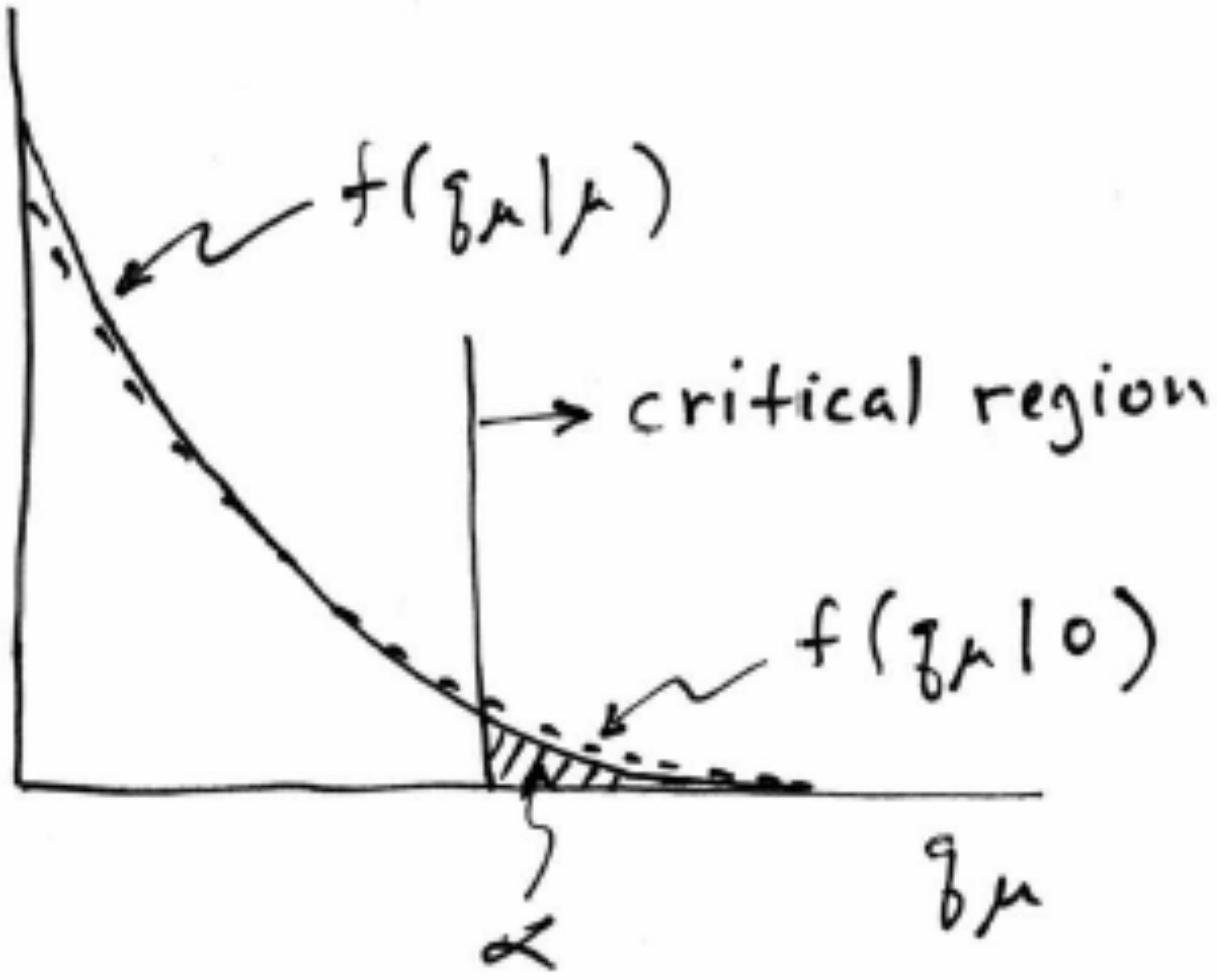
- **In contrast, for a high-sensitivity test, the two pdf's --** $f(q_\mu | \mu)$ and $f(q_\mu | 0)$ **-- are well separated**



In this case, the power is substantially higher than 1–α.
Use this 'power' as a measure of the sensitivity.

low sensitivity & spurious exclusion

- **Consider again the case of low-sensitivity**



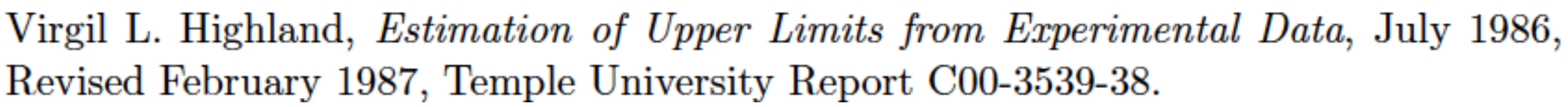
This means that with probability of around $\alpha = 5\%$ (slightly higher), one excludes hypotheses to which one has essentially no sensitivity (e.g., $m_{\mathrm{H}} = 1000$ TeV).

"Spurious exclusion"

spurious = not being what it claims to be

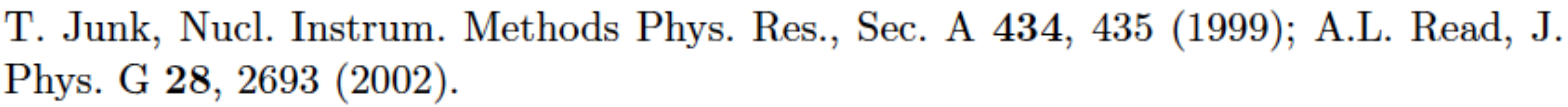
how to handle spurious exclusion

The problem of excluding parameter values to which one has no sensitivity known for a long time; see e.g.,



Virgil L. Highland, *Estimation of Upper Limits from Experimental Data*, July 1986, Revised February 1987, Temple University Report C00-3539-38.

In the 1990s this was re-examined for the LEP Higgs search by Alex Read and others



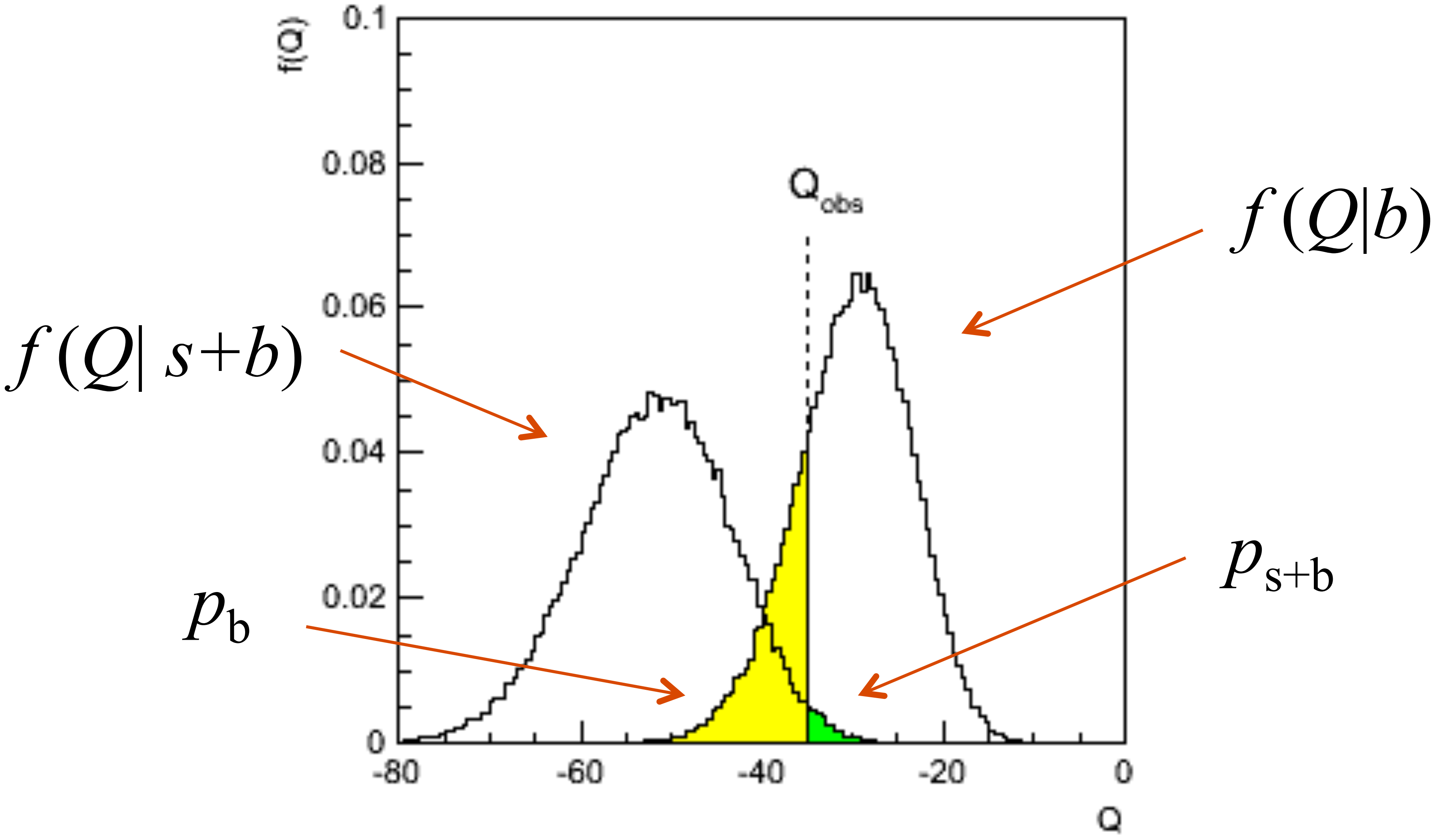
T. Junk, Nucl. Instrum. Methods Phys. Res., Sec. A **434**, 435 (1999); A.L. Read, J. Phys. G **28**, 2693 (2002).

and led to the "CL_s" procedure for upper limits.

Unified intervals also effectively reduce spurious exclusion by the particular choice of critical region.

The $\mathrm{CL_s}$ procedure

In the usual formulation of CL_s, one tests both the $\mu = 0$ (b) and $\mu > 0$ ($\mu s+b$) hypotheses with the same statistic $Q = -2\ln L_{s+b}/L_b$:



The $\mathrm{CL_s}$ procedure

The $\mathrm{CL_s}$ solution (A. Read et al.) is to base the test not on the usual *p*-value ($\mathrm{CL_{s+b}}$), but rather to divide this by $\mathrm{CL_b}$ (~ one minus the *p*-value of the *b*-only hypothesis), i.e.,

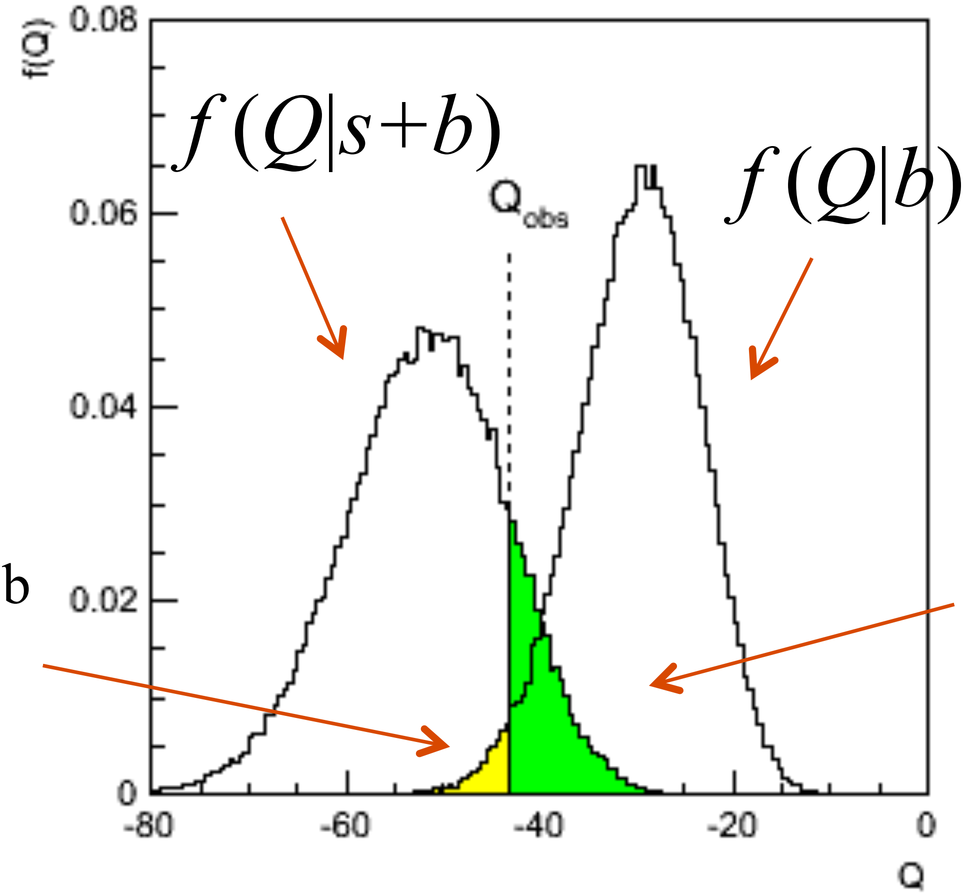
Define:

$$\mathrm{CL_s} = \frac{\mathrm{CL_{s+b}}}{\mathrm{CL_b}}$$

$$= \frac{p_{s+b}}{1 - p_b}$$

Reject s+b hypothesis if:

$$\mathrm{CL_s} \leq \alpha$$



$1-\mathrm{CL_b}$
$= p_\mathrm{b}$

$\mathrm{CL_{s+b}}$
$= p_\mathrm{s+b}$

Reduces "effective" *p*-value when the two distributions become close (prevents exclusion if sensitivity is low).

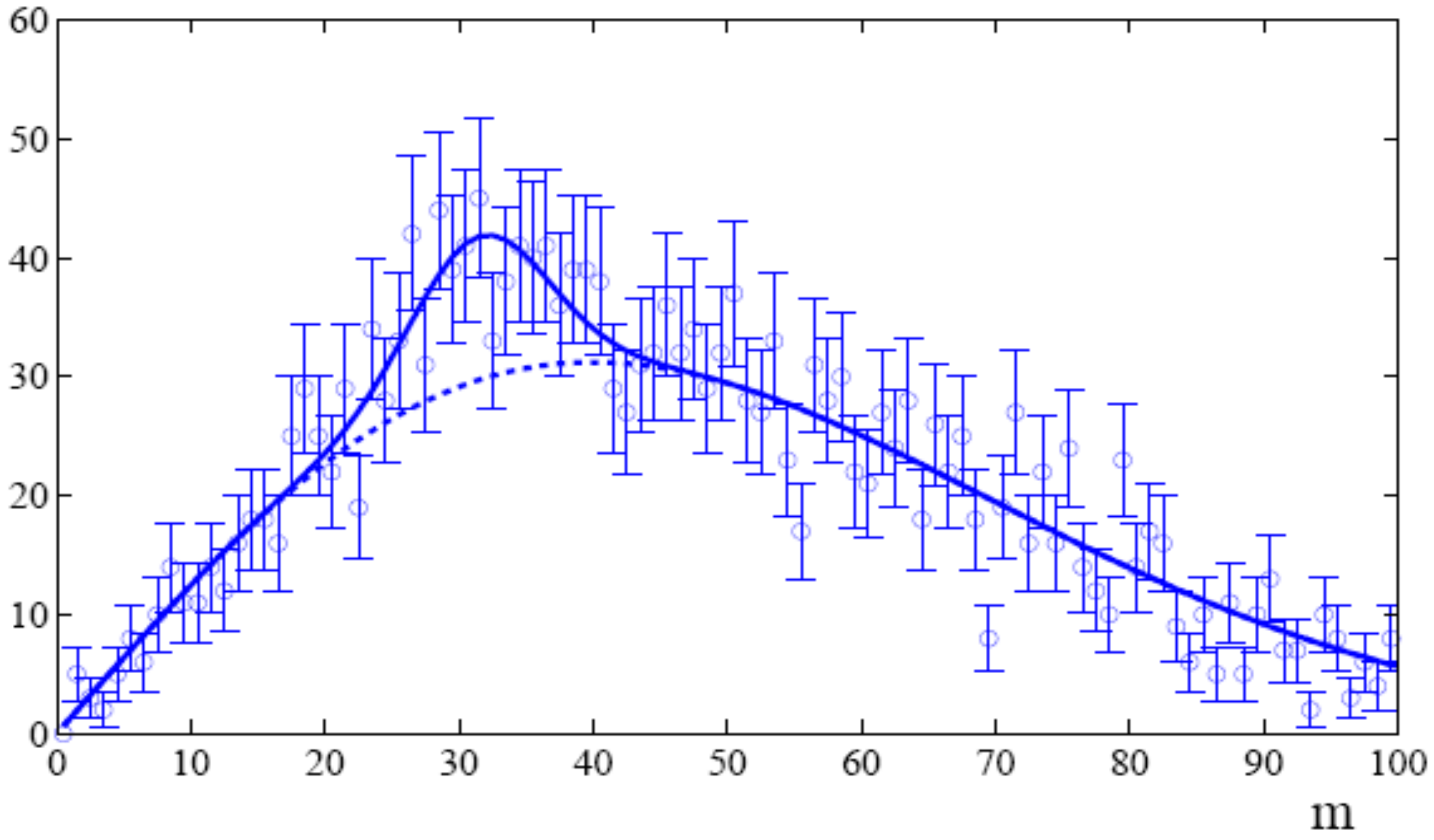
the Look Elsewhere Effect

Gross and Vitells, EPJC 70:525-530,2010, arXiv:1005.1891

The Look-Elsewhere Effect

Suppose a model for a mass distribution allows for a peak at a mass m with amplitude μ.

The data show a bump at a mass m_0.



How consistent is this with the no-bump ($\mu = 0$) hypothesis?

Local p-value

First, suppose the mass m_0 of the peak was specified a priori.

Test consistency of bump with the no-signal ($\mu = 0$) hypothesis with e.g. likelihood ratio

$$t_{\rm fix} = -2 \ln \frac{L(0, m_0)}{L(\hat{\mu}, m_0)}$$

where "fix" indicates that the mass of the peak is fixed to m_0.

The resulting p-value

$$p_{\rm local} = \int_{t_{\rm fix,obs}}^{\infty} f(t_{\rm fix}|0)\, dt_{\rm fix}$$

gives the probability to find a value of $t_{\rm fix}$ at least as great as observed at the specific mass m_0 and is called the local p-value.

Global *p*-value

But suppose we did not know where in the distribution to expect a peak.

What we want is the probability to find a peak at least as significant as the one observed anywhere in the distribution.

Include the mass as an adjustable parameter in the fit, test significance of peak using

$$t_{\text{float}} = -2\ln\frac{L(0)}{L(\hat{\mu},\hat{m})}$$

(Note m does not appear in the $\mu = 0$ model.)

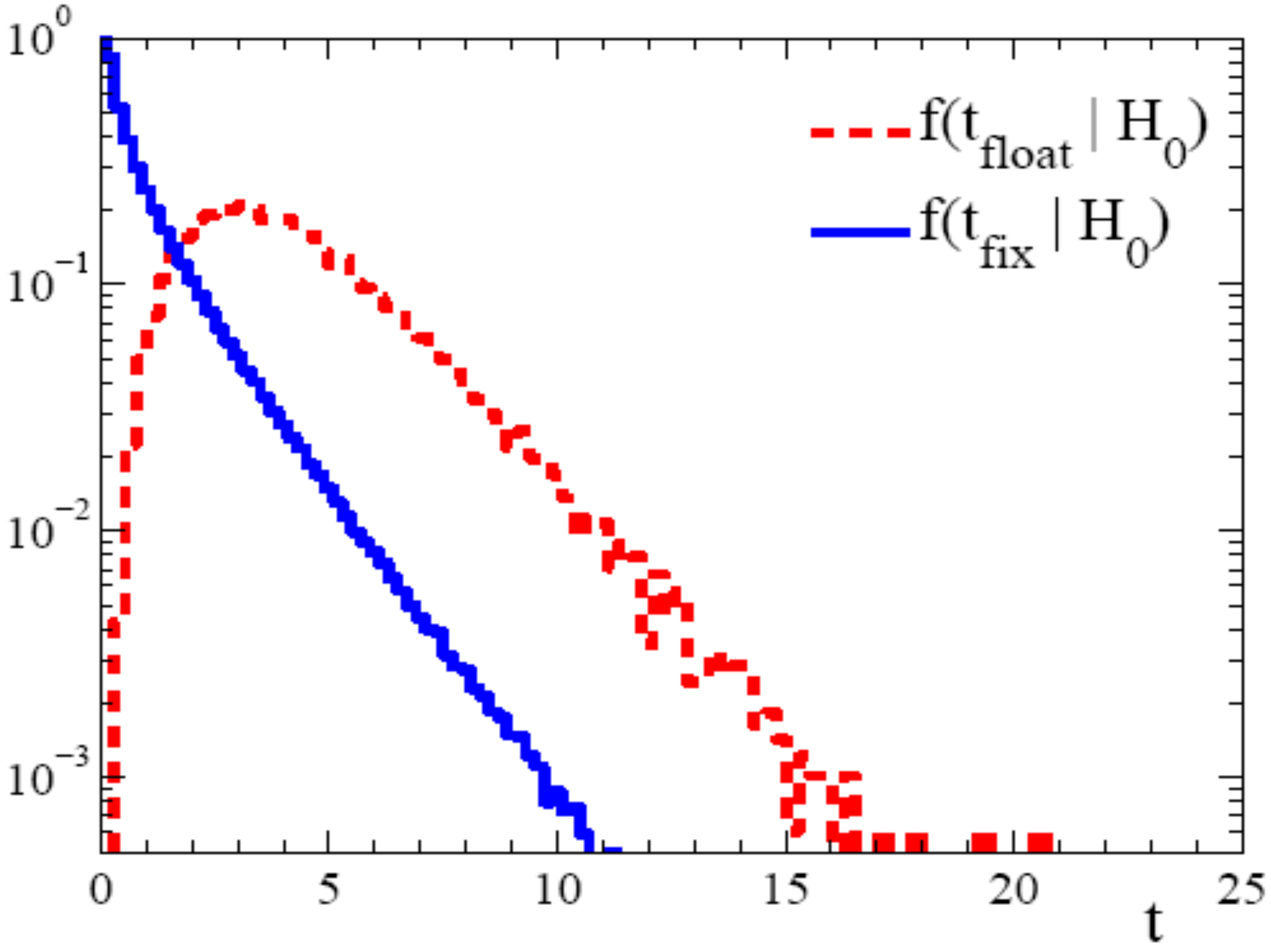
$$p_{\text{global}} = \int_{t_{\text{float,obs}}}^{\infty} f(t_{\text{float}}|0)\,dt_{\text{float}}$$

Gross and Vitells

Distributions of t_{fix}, t_{float}

For a sufficiently large data sample, t_{fix} ~chi-square for 1 degree of freedom (Wilks' theorem).

For t_{float} there are two adjustable parameters, μ and m, and naively Wilks theorem says $t_{\mathrm{float}} \sim$ chi-square for 2 d.o.f.



In fact Wilks' theorem does not hold in the floating mass case because on of the parameters (m) is not-defined in the $\mu = 0$ model.

So getting t_{float} distribution is more difficult.

Gross and Vitells

Approximate correction for LEE

We would like to be able to relate the *p*-values for the fixed and floating mass analyses (at least approximately).

Gross and Vitells show the *p*-values are approximately related by

$$p_{\text{global}} \approx p_{\text{local}} + \langle N(c) \rangle$$

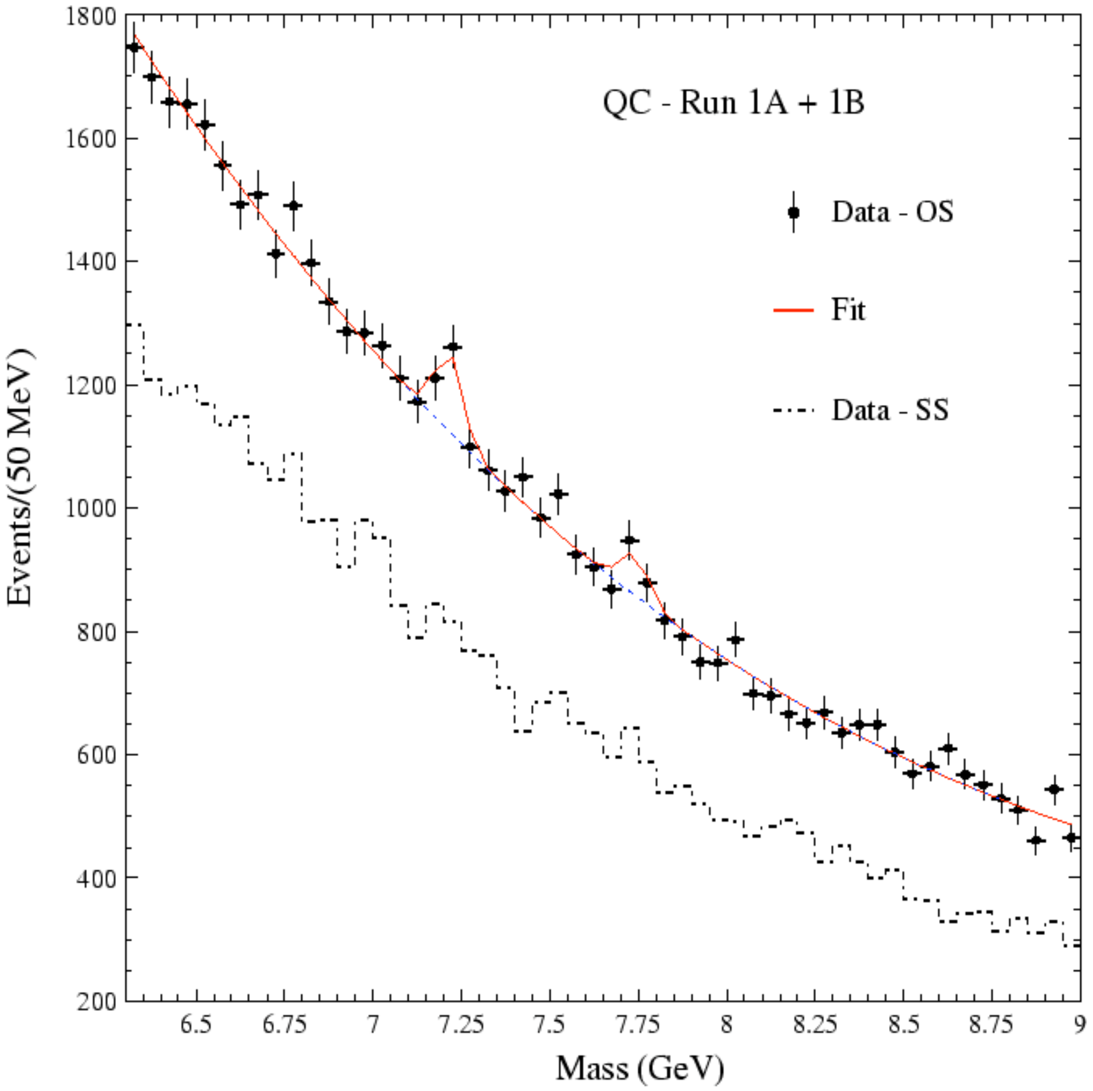
where $\langle N(c) \rangle$ is the mean number "upcrossings" of $-2\ln L$ in the fit range based on a threshold

$$c = t_{\text{fix}} = Z_{\text{local}}^2$$

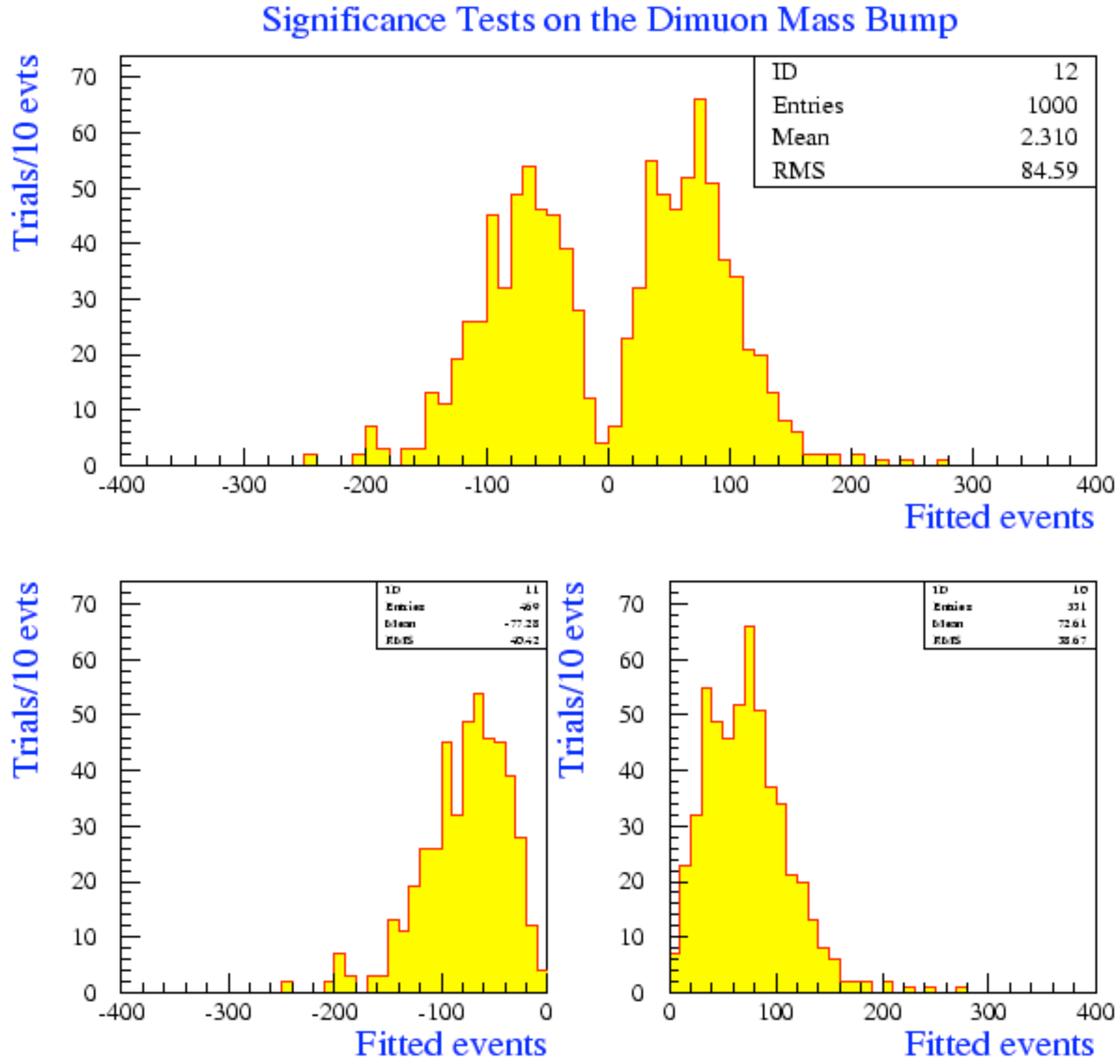
and where $Z_{\text{local}} = \Phi^{-1}(1 - p_{\text{local}})$ is the local significance.

So we can either carry out the full floating-mass analysis (e.g. use MC to get *p*-value), or do fixed mass analysis and apply a correction factor (much faster than MC).

An internal CDF study that didn’t make it to prime time – dimuon mass spectrum with signal fit (not enough PE’s)



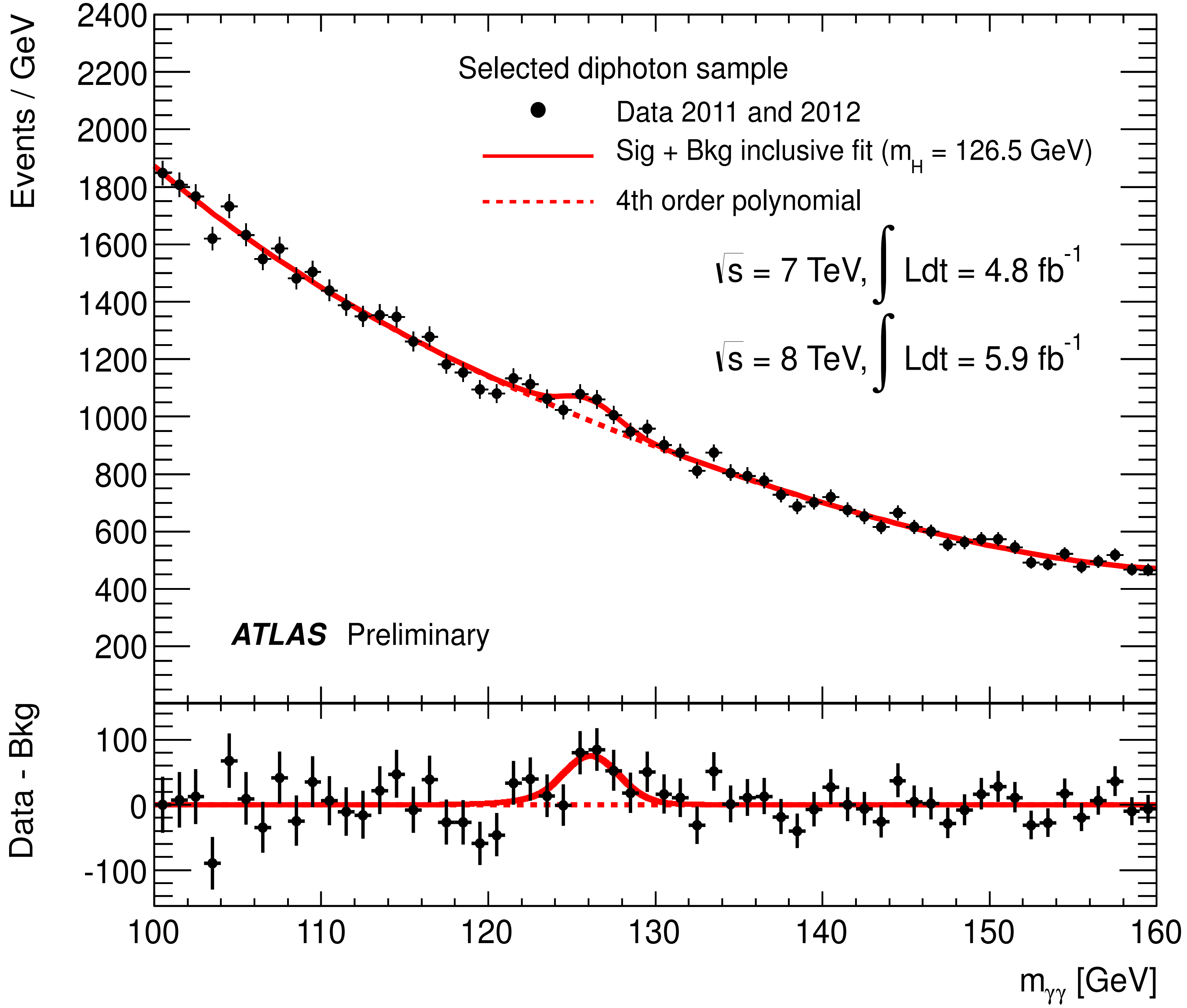
249.7±60.9 events fit in bigger signal peak (4σ? No!)



Null hypothesis pseudoexperiments with largest peak fit values

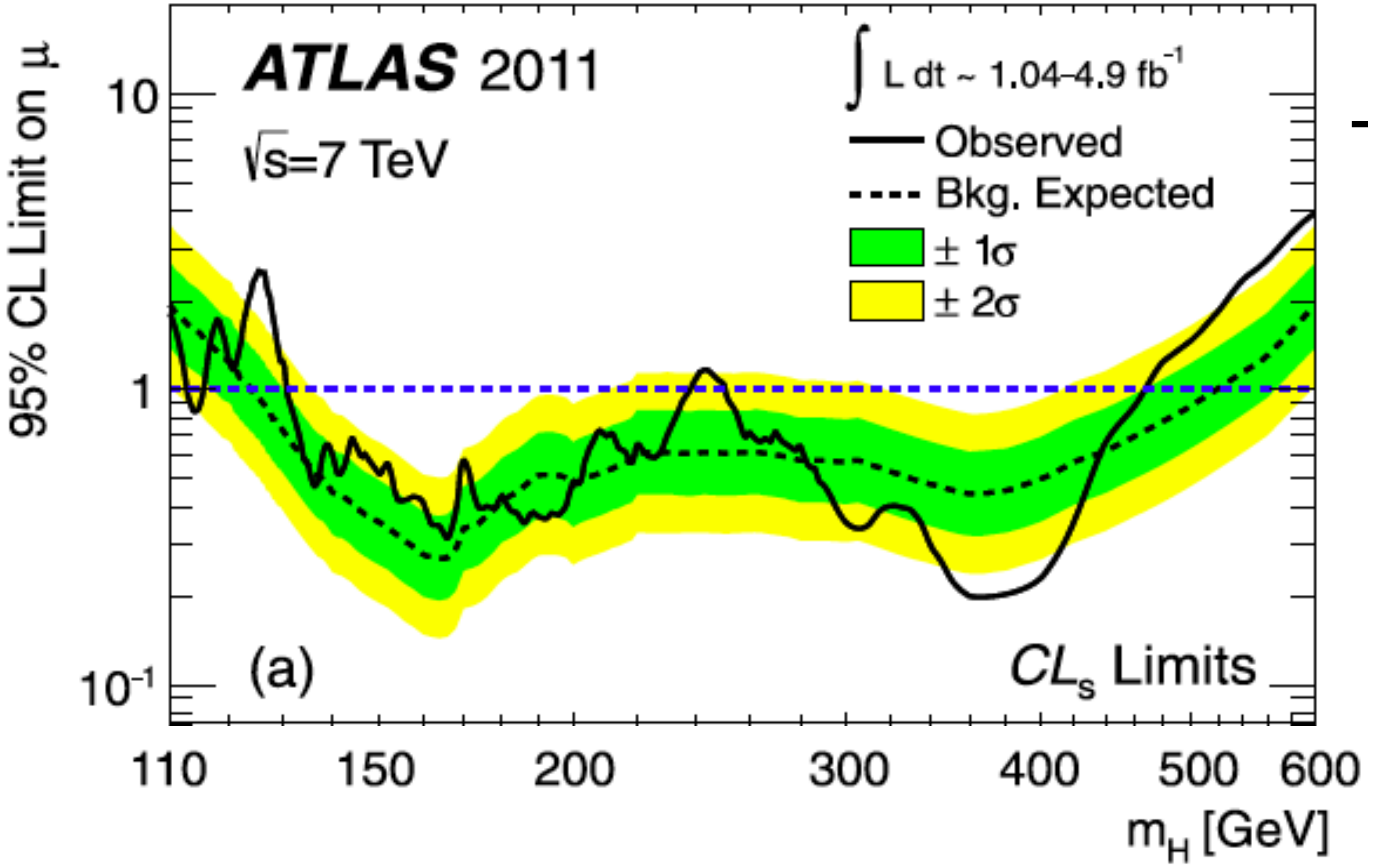
Examples to test what you’ve learned

what to make sense of m_H plots, statistically



how to read the green & yellow plots

- For every (assumed) value of m_H, we want to find the CL$_s$ upper limit on $\mu \equiv \sigma(H)/\sigma_{\rm SM}(H)$ (*solid curve*)
- Also shown is the ‘expected upper limit’, determined for each assumed m_H value, under the assumption that we see no excess above background.

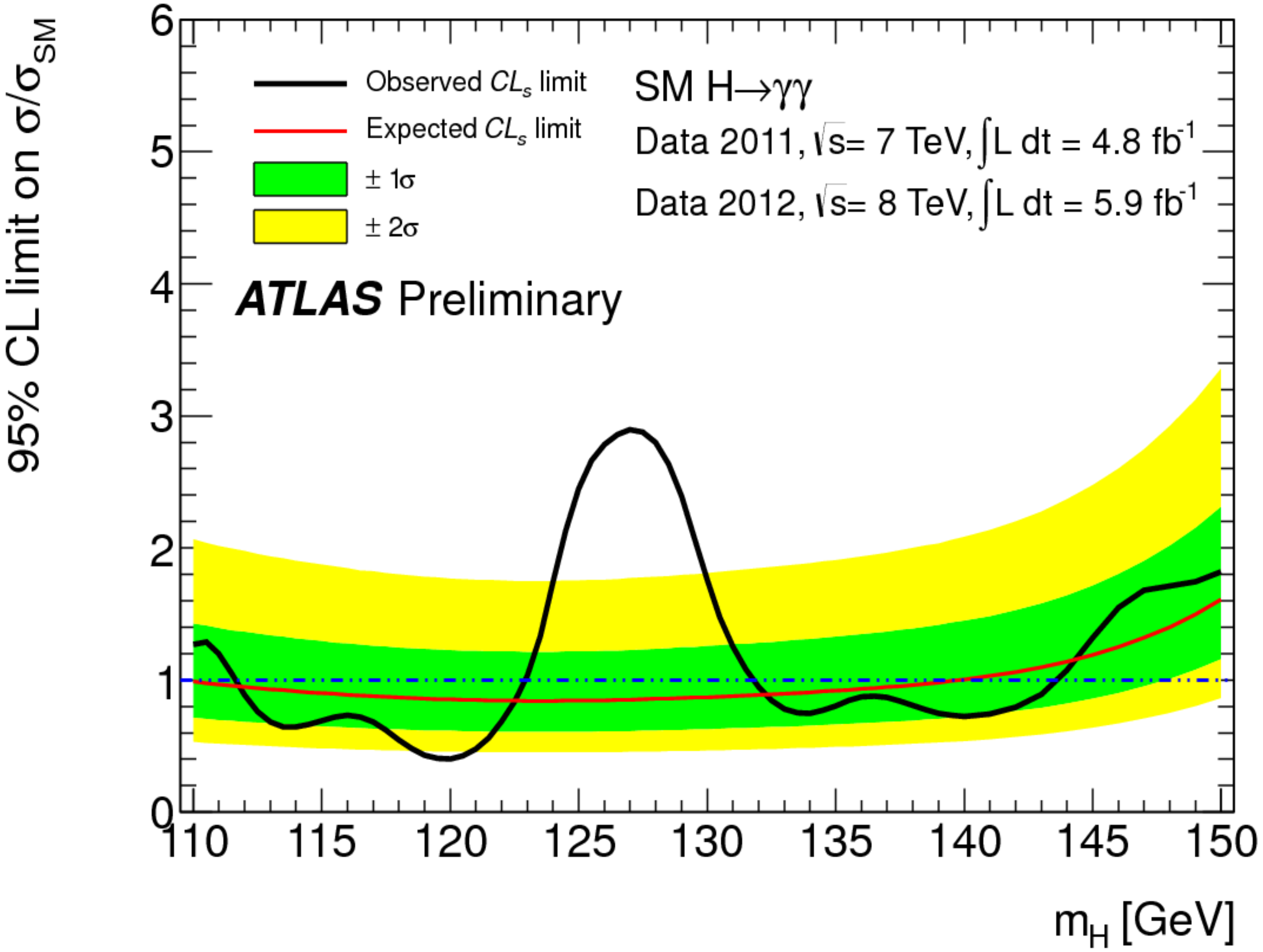


--- median expected limit w/o signal

ATLAS, Phys. Lett. B 710 (2012) 49-66

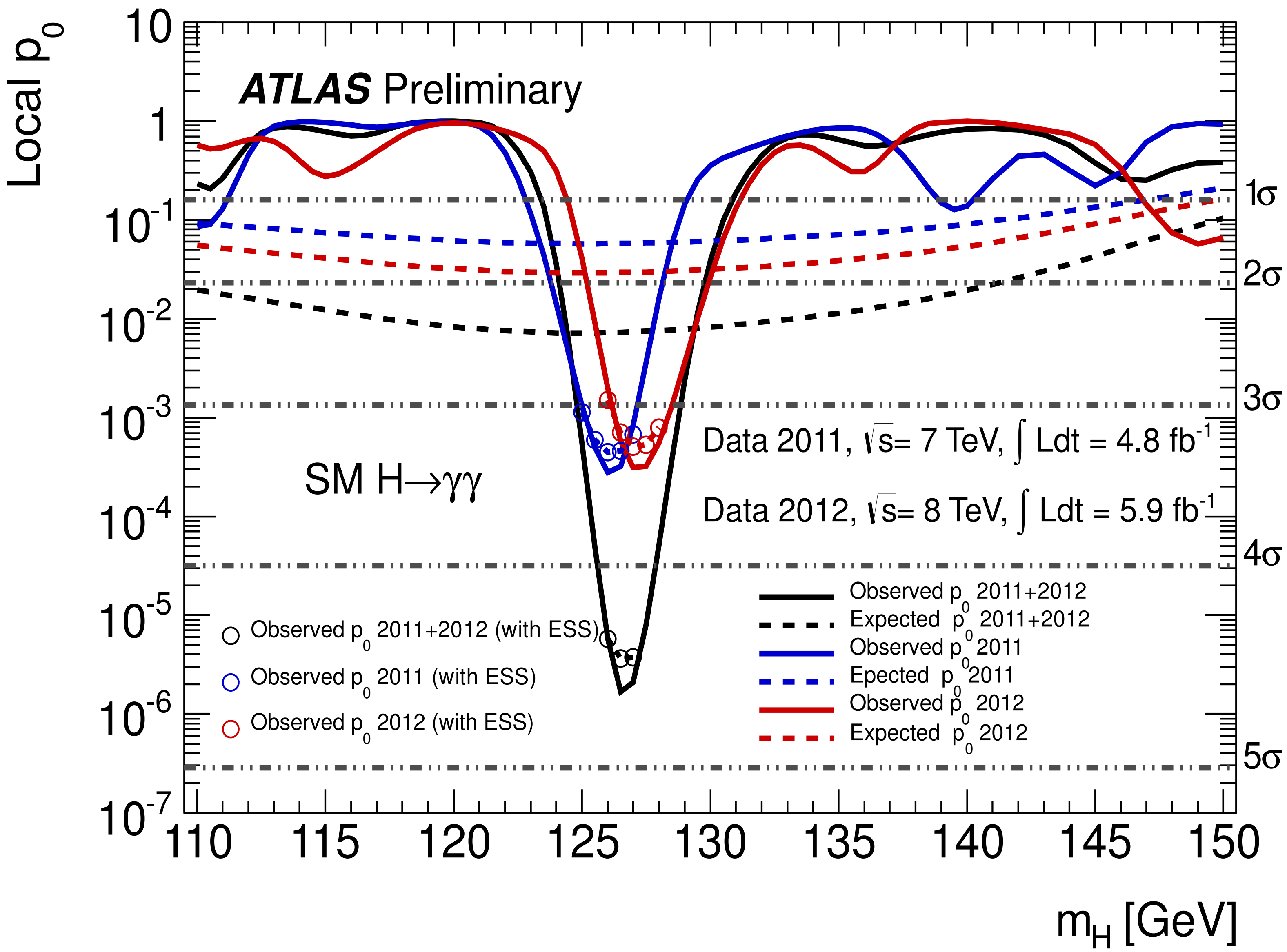
how to read the green & yellow plots

- For every (assumed) value of m_H, we want to find the CL$_s$ upper limit on $\mu \equiv \sigma(H)/\sigma_{\rm SM}(H)$ (*solid curve*)
- Also shown is the ‘expected upper limit’, determined for each assumed m_H value, under the assumption that we see no excess above background.



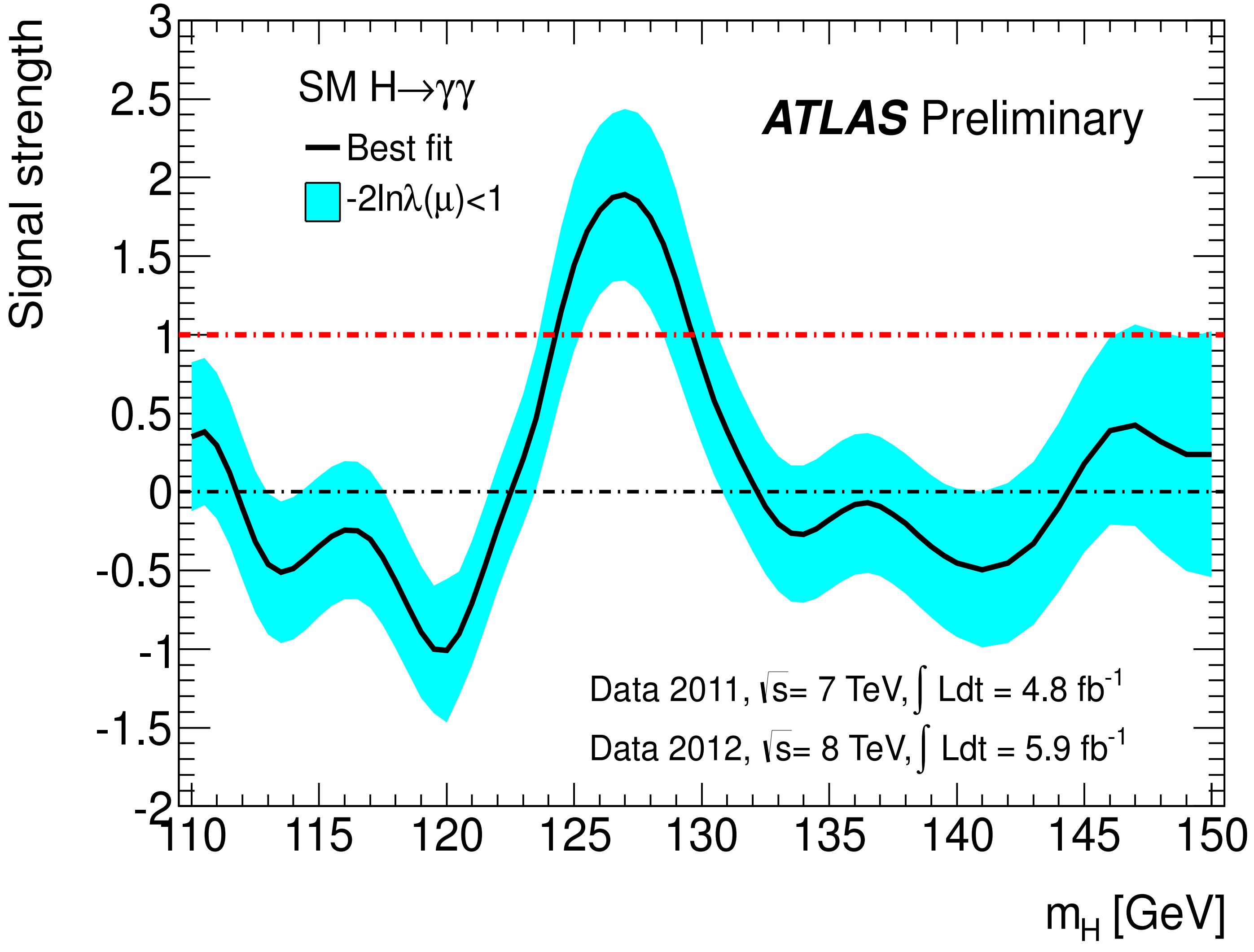
how to read the p_0 plots

- The **local** p_0 values for a SM Higgs boson as a function of assumed m_H.
- The minimal p_0 (observed) is 2×10^{-6} at $m_H = 126.5$ GeV.
  $\Rightarrow$ local significance of $4.7\sigma \rightarrow$ reduced to 3.6σ after LEE



how to read the "*blue band*" plots

- $\hat{\mu}$ vs. m_H where $\hat{\mu}$ is the signal strength ($= \sigma/\sigma_{\mathrm{SM}}$) estimated by likelihood method[1]. The blue band corresponds to approx. $\pm 1\sigma$ error bar for μ.



---

[1] Some details are skipped, for the sake of simplicity

Now that you have the language
to talk about stat. interpretation of HEP
results (e.g. LHC),
it's your job to explore & enjoy them!

Thank you!