Entanglement Witness of Primordial Perturbation

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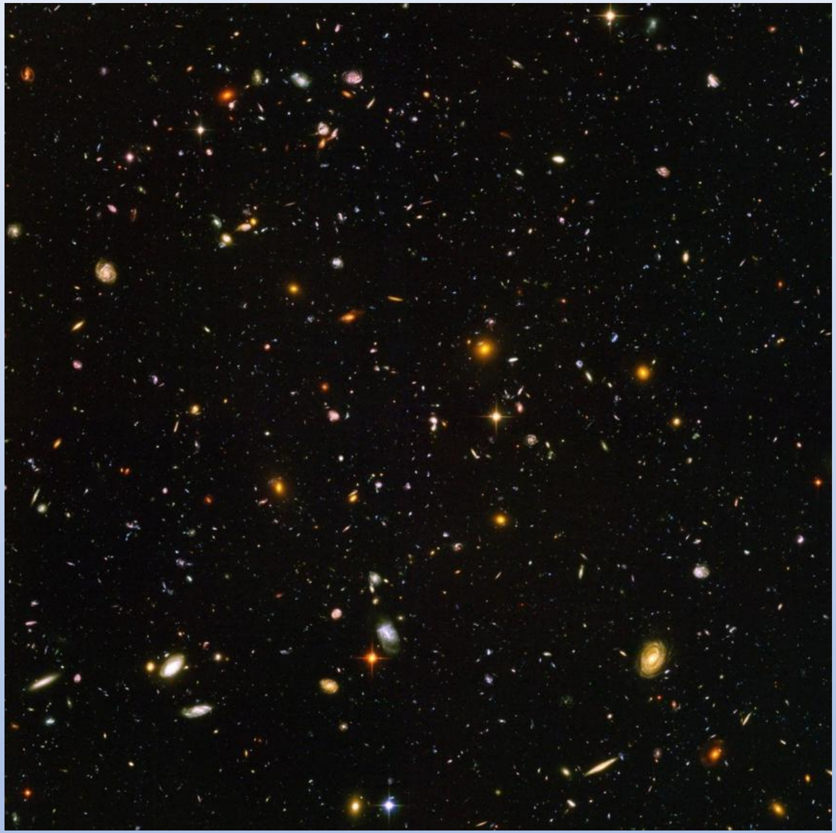
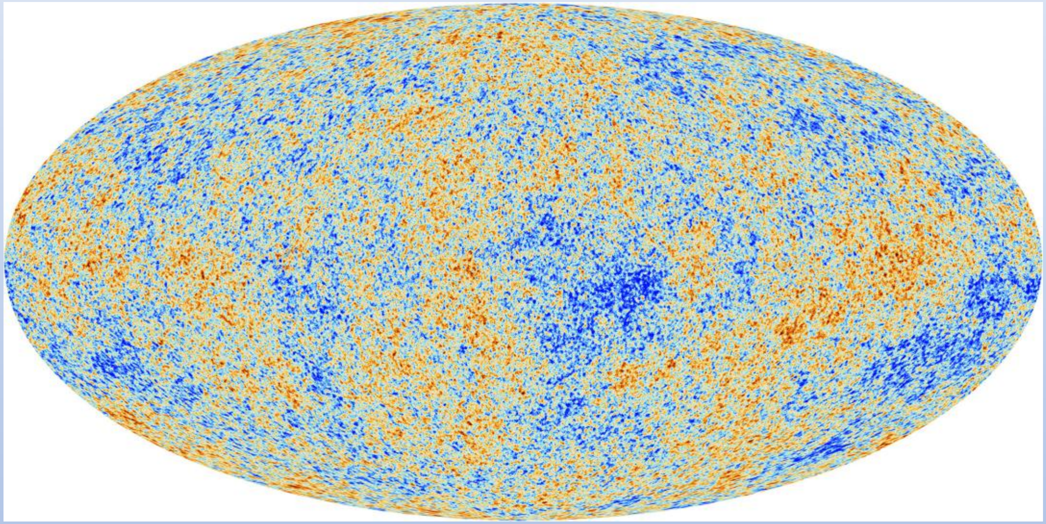
Primordial perturbation



https://science.nasa.gov/universe/galaxies/large-scale-structures/

Large Scale Structure



https://www.esa.int/ESA_Multimedia/Images/2013/03/Planck_CMB

Temperature Anisotropy of CMB

Inflation

Caused by a scalar field, called inflaton

Lagrangian for inflaton field

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$$

EOM for inflaton field in homogeneous and isotropic metric

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

The second term represents friction term

If $\dot{\phi}$ is small, the inflaton field moves slowly

$$\dot{\phi} \approx -\frac{1}{3H} \frac{dV}{d\phi}$$

Slow-roll conditions

- $\dot{\phi}^2 \ll 2V$
- $|\ddot{\phi}| \ll 3H|\dot{\phi}|$

I Quantization of Primordial Perturbation

Einstein-Hilbert action

$$S = \frac{1}{16\pi G}\int R\sqrt{-g}\,d^4x$$

Einstein Hilbert action + Scalar field action

$$S = \int \sqrt{-g}\,d^4x\left[\frac{R}{16\pi G} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

The perturbation for a scalar field

$$\phi(x,\eta) = \bar{\phi}(\eta) + \delta\phi(x,\eta)$$

Background (→ $\bar{\phi}(\eta)$)

perturbed (→ $\delta\phi(x,\eta)$)

I Quantization of Primordial Perturbation

Perturbing the Einstein-Hilbert action up to the second order.

$$S = \frac{1}{2}\int_{\boldsymbol{R}^3} d^4x \left[(v')^2 - \delta^{ij}\partial_i v \partial_j v + \frac{\left(a\sqrt{\epsilon_1}\right)''}{a\sqrt{\epsilon_1}} v^2 \right]$$

v : Mukhanov-Sasaki variable

$$v = a\left(\delta\phi - \frac{\bar{\phi}'}{\mathcal{H}} D\right)$$

a : scale factor

ϵ_1 : slow-roll parameter

Fourier transform of v

$$v(\eta, x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\boldsymbol{R}^3} d^3k \;\; v_k(\eta) e^{ik\cdot x}$$

$$S = \frac{1}{2}\int_{\boldsymbol{R}^3} d\eta d^3k \left[v_k' v_k^{*\prime} - \frac{z'}{z}\left(v_k' v_k^* + v_k v_k^{*\prime}\right) + \left(\frac{z^{2\prime}}{z^2} - k^2\right) v_k v_k^* \right]$$

$z = aM_{Pl}\sqrt{2\epsilon_1}$

M_{Pl} : Plank mass

I Quantization of Primordial Perturbation

the conjugate momentum of v_k

$$p_k = \frac{\partial L}{\partial v_k^{*\prime}} = v_k' - \frac{z'}{z} v_k$$

Hamiltonian

$$H = \int_{\boldsymbol{R}^3} d^3k \left(p_k v_k' - L\right)$$

$$\hat{v}_{-k} = \hat{v}_k^{\dagger}$$

creation and annihilation operators.

$$\hat{v}_k = \frac{1}{\sqrt{2k}}\left(\hat{c}_k + \hat{c}_{-k}^{\dagger}\right) \qquad \hat{p}_k = -i\sqrt{\frac{k}{2}}\left(\hat{c}_k - \hat{c}_{-k}^{\dagger}\right)$$

Hamiltonian operator

$$\hat{H} = \int_{\boldsymbol{R}^3} d^3k \left[\frac{k}{2}\left(\hat{c}_k\hat{c}_k^{\dagger} + \hat{c}_{-k}\hat{c}_{-k}^{\dagger}\right) - \frac{i}{2}\frac{z'}{z}\left(\hat{c}_k\hat{c}_{-k} - \hat{c}_{-k}^{\dagger}\hat{c}_k^{\dagger}\right)\right]$$

harmonic oscillator

Inverted harmonic oscillator

Ⅰ Quantization of Primordial Perturbation

<inverted harmonic oscillator>

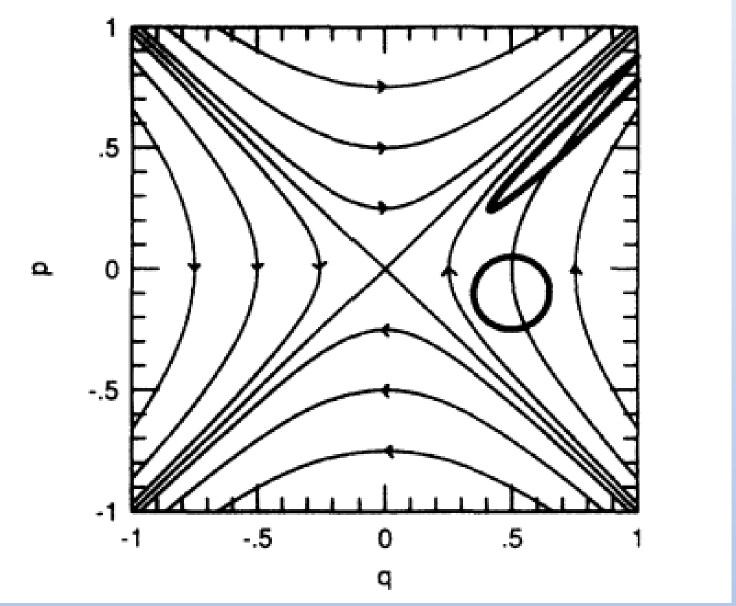
Andreas Albrecht, Pedro Ferreira, Michael Joyce, Tomislav Prokopec[9303001]

Hamiltonian

$$H = \frac{q^2}{2} - \frac{p^2}{2}$$

$$q(t) = Ae^{t} + Be^{-t}$$
$$p(t) = Ae^{t} - Be^{-t}$$

A,B : constant of integration



Phase space trajectories for a classical inverted harmonic oscillator

the circle evolves into the squeezed shape

Ⅰ Quantization of Primordial Perturbation

Heisenberg equation

$$\frac{d\hat{c}_k}{d\eta} = -i\left[\hat{c}_k, \widehat{H}\right] \quad \Longleftrightarrow \quad i\frac{d\hat{c}_k}{d\eta} = k\hat{c}_k + i\frac{z'}{z}\hat{c}^{\dagger}_{-k}$$

Bogoliubov transformation

$$\hat{c}_k(\eta) = \mathrm{u_k}(\eta)\hat{c}_k(\eta_{ini}) + \mathrm{v_k}(\eta)\,\hat{c}^{\dagger}_{-k}(\eta_{ini}) \quad \dots \ (A)$$

$$\Downarrow$$

$$i\frac{du_k(\eta)}{d\eta} = ku_k(\eta) + i\frac{z'}{z}v^*_k(\eta)$$

$$i\frac{dv_k(\eta)}{d\eta} = kv_k(\eta) + i\frac{z'}{z}u^*_k(\eta)$$

Ⅰ Quantization of Primordial Perturbation

I want to consider the evolution from vacuum state.
The state is described by acting $\hat{c}_k$, $\hat{c}_{-k}$, $\hat{c}_k^\dagger$ and $\hat{c}^\dagger_{-k}$ on vacuum state.
But when the Hamiltonian is described by the form of an inverted harmonic oscillator, the evolution is written by the following operators.

Introduce two operators

①two-mode squeezing operator $\hat{S}(r_k, \phi_k)$

$$\hat{S}(r_k, \phi_k) = e^{\hat{B}_k}$$

$$\hat{B}_k \equiv r_k e^{-2i\phi_k} \hat{c}_{-k}(\eta_{ini})\, \hat{c}_k(\eta_{ini}) - r_k\, e^{2i\phi_k} \hat{c}^\dagger_{-k}(\eta_{ini})\, \hat{c}_k^\dagger(\eta_{ini})$$

②rotation operator $\hat{R}(\theta_k)$

$$\hat{R}(\theta_k) = e^{\hat{D}_k}$$

$$\hat{D} \equiv -i\theta_k \hat{c}_k^\dagger(\eta_{ini})\hat{c}_k(\eta_{ini}) - i\theta_k \hat{c}^\dagger_{-k}(\eta_{ini})\hat{c}_{-k}(\eta_{ini})$$

Ⅰ Quantization of Primordial Perturbation

Calculate the quantity $\hat{R}_k^\dagger\,\hat{S}_k^\dagger \hat{c}_k(\eta_{ini})\hat{S}_k\,\hat{R}_k$

$$\hat{R}_k^\dagger\,\hat{S}_k^\dagger \hat{c}_k(\eta_{ini})\hat{S}_k\,\hat{R}_k = e^{-i\theta_{k,1}}\cosh r_k\,\hat{c}_k(\eta_{ini}) - e^{-i\theta_{k,2}+2i\phi_k}\sinh\,r_k\,\hat{c}_{-k}^\dagger(\eta_{ini}) \quad \cdots \text{(B)}$$

By comparing equation (A) and (B)

$$u_k(\eta) = e^{-i\theta_k}\cosh r_k$$

$$v_k(\eta) = -e^{-i\theta_k+2i\phi_k}\sinh r_k$$

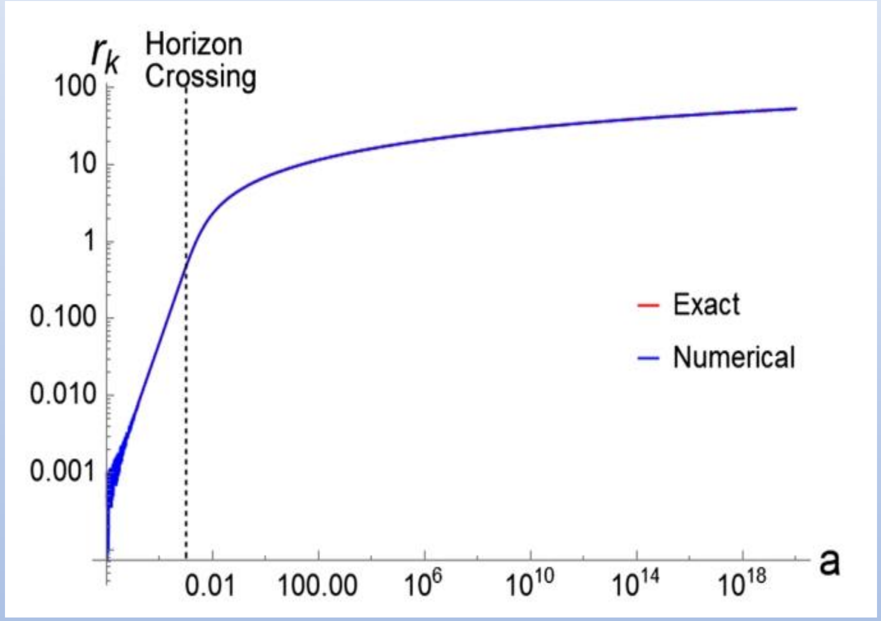
Heisenberg equation of $u_k(\eta)$, $v_k(\eta)$

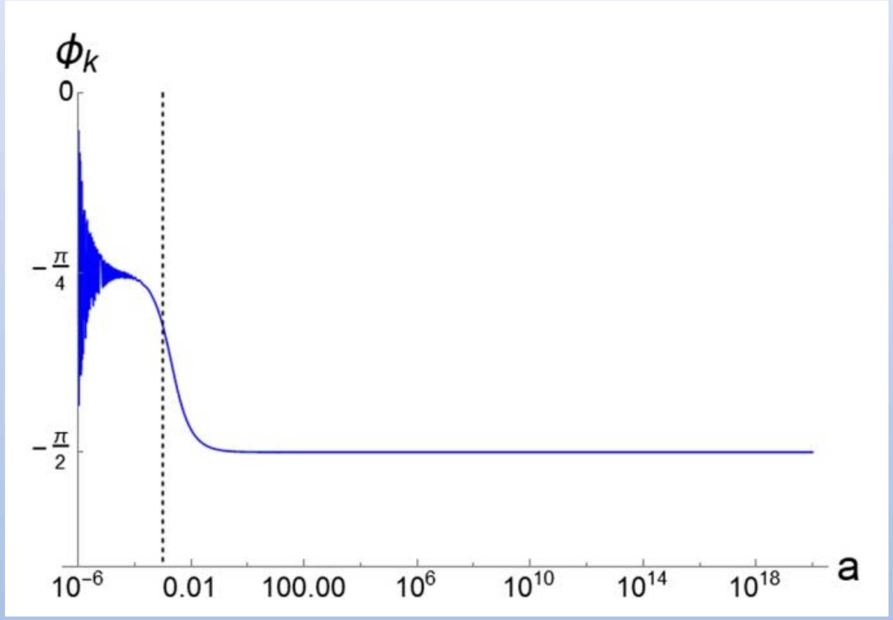
$$\frac{dr_k}{d\eta} = -\frac{z'}{z}\cos(2\phi_k)$$

$$\frac{d\phi_k}{d\eta} = -k + \frac{z'}{z}\coth(2r_k)\sin(2\phi_k)$$

$$\frac{d\theta_k}{d\eta} = k + \frac{z'}{z}\tanh r_k\sin(2\phi_k)$$

I Quantization of Primordial Perturbation





Arpan Bhattacharyya, Saurya Das, S. Shajidul Haque, Bret [2001.08664]

I Quantization of Primordial Perturbation

From $|0_k, 0_{-k}\rangle$ into two-mode squeezed state

$$\hat{S}(r_k, \phi_k)\hat{R}(\theta_k)|0_k, 0_{-k}\rangle = \hat{S}(r_k, \phi_k)|0_k, 0_{-k}\rangle$$

By using operator ordering theorem

$$\hat{S}(r_k, \phi_k)\hat{R}(\theta_k)|0_k, 0_{-k}\rangle = \frac{1}{\cosh r_k}\sum_{n=0}^{\infty} e^{2in\phi_k}(-1)^n \tanh r_k |n_k, n_{-k}\rangle$$

Density matrix for two-mode squeezed state

$$\hat{\rho}(k, -k) = \sum_{n,n'=0}^{\infty} e^{2i(n-n')\phi_k}(-1)^{n+n'} \tanh^{n+n'} r_k \, |n_k, n_{-k}\rangle\langle n'_k, n'_{-k}|$$

Ⅱ Entanglement Witness

Separable state

$$|\Psi\rangle_{AB} = \sum_{j,k} a_j b_k \, |\psi_j\rangle_A \otimes |\phi_k\rangle_B$$

Subsystem A and B can be measured independently.

Entangled state

$$|\Psi\rangle_{AB} = \sum_{j} c_j \, |\psi_j\rangle_A \otimes |\phi_j\rangle_B$$

Subsystem A and B can’t be measured independently

Ⅱ Entanglement Witness

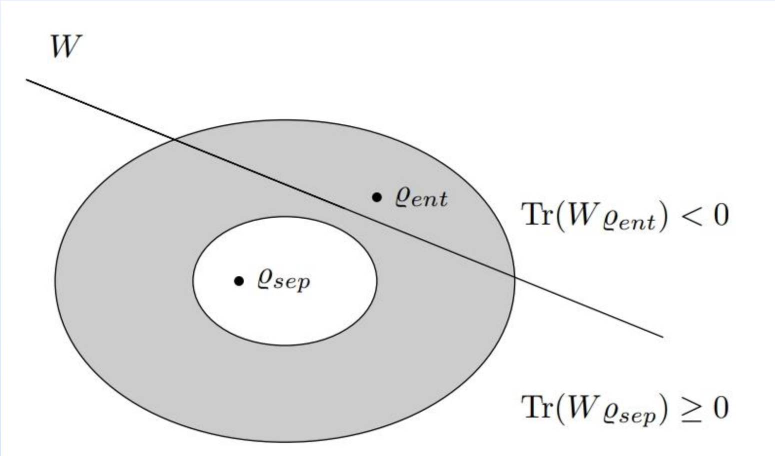
Conditions of witness operator

(1) For all separable states

$$Tr[\rho_{sep}W] \geq 0$$

(2) For at least one entangled states

$$Tr[\rho_{ent}W] < 0$$



By computing this quantity, We can confirm whether the states is entangled or not.

Horodecki et al [0702225]

Ⅱ Entanglement Witness

The condition of witness operator

(1)Non-negativity of mean value

$$Tr[W] \geq 0$$

(2)Negative eigenvalue

$$W = \frac{1}{d}\mathbf{1} - |\psi\rangle\langle\psi|$$

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} |m_k, m_{-k}\rangle$$

Ⅱ Entanglement Witness

Witness operator for 2 level

$$W = \frac{1}{2}\mathbf{1} - |\psi\rangle\langle\psi|$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\sum_{m=0}^{1}|m_k, m_{-k}\rangle = \frac{1}{\sqrt{2}}(|0_k, 0_{-k}\rangle + |1_k, 1_{-k}\rangle)$$

⇒ Bell state

||

Maximal entangled state

$$|\psi_{AB}^{max}\rangle = \frac{1}{\sqrt{N}}\sum_{m=1}^{N}|\psi_m\rangle_A \otimes |\psi_m\rangle_B$$

Bell state density matrix

$$\hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2}(|0_k, 0_{-k}\rangle + |1_k, 1_{-k}\rangle)(\langle 0_k, 0_{-k}| + \langle 1_k, 1_{-k}|)$$

Ⅱ Entanglement Witness

Confirm whether the state is entangled or not

$$Tr[\rho W] = Tr\left[\rho\left\{\frac{1}{2}\mathbf{1} - |\psi\rangle\langle\psi|\right\}\right]$$

$Tr[\rho] = 1$ ➡ $$= \frac{1}{2}Tr[\rho] - Tr[\rho|\psi\rangle\langle\psi|]$$

$Tr[\rho^2] = Tr[\rho]$ ➡ $$= \frac{1}{2} - Tr[\rho^2]$$

$$= -\frac{1}{2}$$

⬇

For Bell state, entanglement witness is negative

Ⅱ Entanglement Witness

Entanglement witness for two-mode squeezed state

$$Tr[\hat{\rho}W] = \lim_{d\to\infty}\left(\frac{1}{d}Tr[\hat{\rho}] - Tr[\hat{\rho}|\psi\rangle\langle\psi|]\right)$$

$$= \lim_{d\to\infty}\frac{1}{d}\{1 - \frac{1}{\cosh^2 r_k}\sum_{n,n'=0}^{\infty}\sum_{m,m'=0}^{d-1} e^{2i(n-n')\phi_k}(-1)^{n+n'}\tanh^{n+n'} r_k$$
$$\times Tr[|n_k, n_{-k}\rangle\langle n'_k, n'_{-k}|m_k, m_{-k}\rangle\langle m'_k, m'_{-k}|]\}$$

$$= \lim_{d\to\infty}\frac{1}{d}\{1 - \frac{1}{\cosh^2 r_k}\sum_{n,n'=0}^{\infty}\sum_{m,m'=0}^{d-1}\sum_{l=0}^{\infty} e^{2i(n-n')\phi_k}(-1)^{n+n'}\tanh^{n+n'} r_k$$
$$\times \underbrace{\langle l_k, l_{-k}|n_k, n_{-k}\rangle}_{\delta_{ln}}\underbrace{\langle n'_k, n'_{-k}|m_k, m_{-k}\rangle}_{\delta_{n'm}}\underbrace{\langle m'_k, m'_{-k}|l_k, l_{-k}\rangle}_{\delta_{m'l}}\}$$

Ⅱ Entanglement Witness

$$= \lim_{d\to\infty} \frac{1}{d}\left\{1 - \frac{1}{\cosh^2 r_k}\sum_{n,n\prime=0}^{d-1} e^{2i(n-n')\phi_k}(-1)^{n+n'}\tanh^{n+n'} r_k\right\}$$

Taking superhorizon limit, $\phi_k \to -\frac{\pi}{2}$

$$= \lim_{d\to\infty} \frac{1}{d}\left\{1 - \frac{1}{\cosh^2 r_k}\sum_{n=0}^{d-1}\tanh^n r_k \sum_{n'=0}^{d-1}\tanh^{n'} r_k\right\}$$

Taking $r_k \approx 50$,at the end of inflation, $Tr[\hat{\rho}W]$ is negative

$Tr[\rho W]$ is negative at the end of inflation, so the state for primordial perturbation is entangled.

Ⅲ Future Works

- Build a way to observe quantumness such as Entanglement Witness.
- Consider decoherence in the early universe.