Cosmic History and the Gravitational Waves

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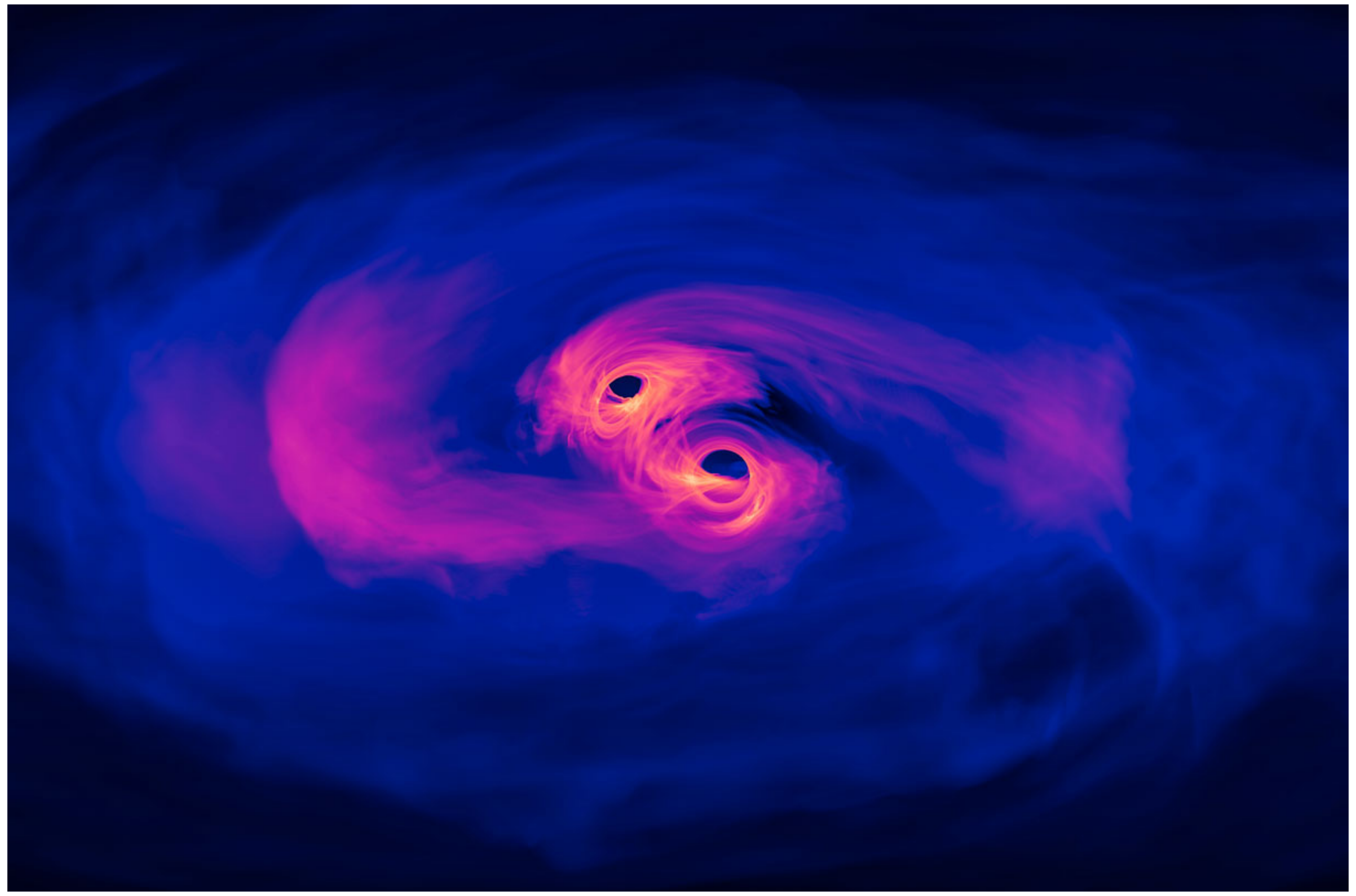
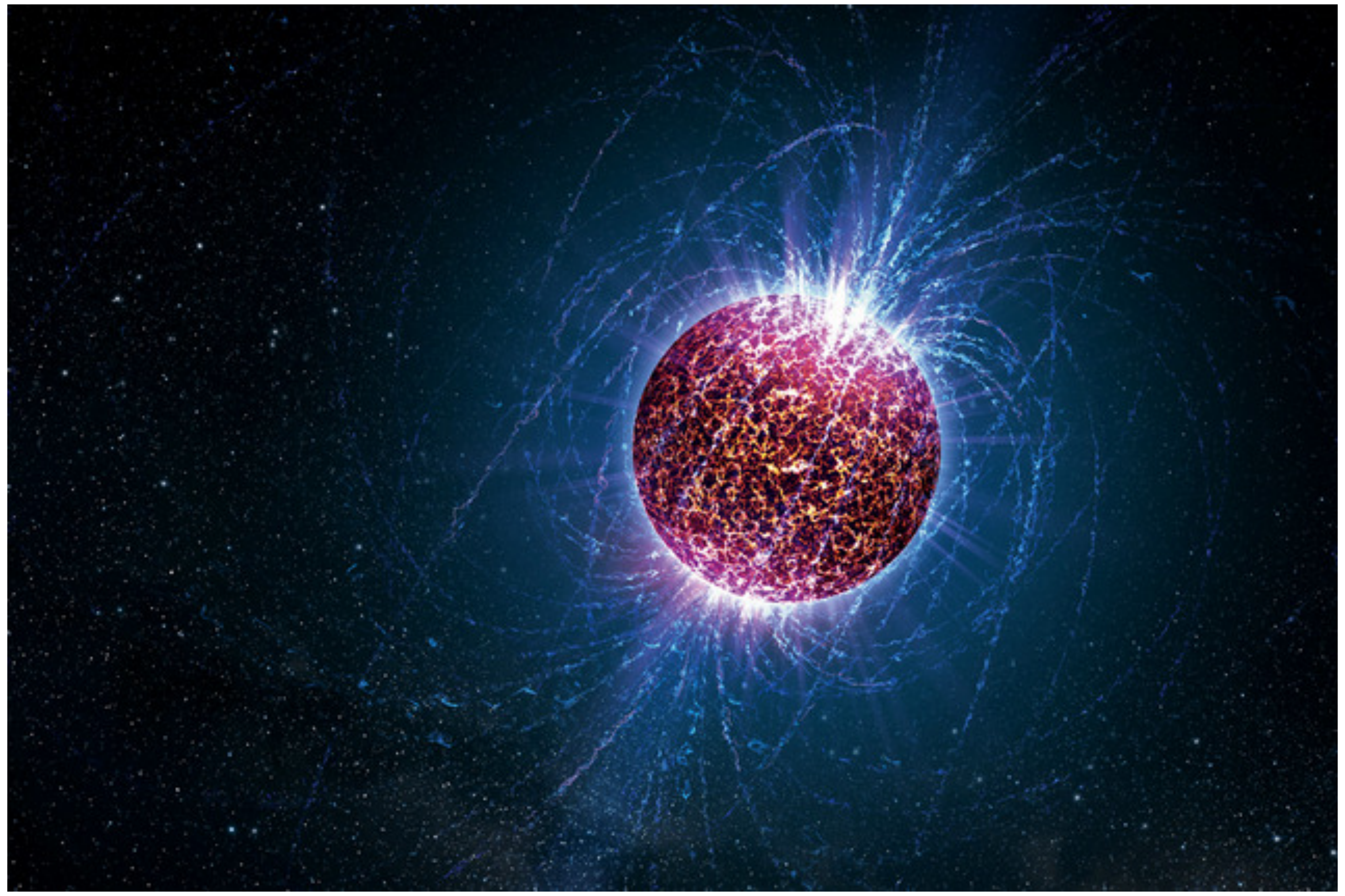
Saga-Yonsei Joint Workshop

Gravitational Waves

- Perturbation in the metric propagating through spacetime
- $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu} < < 1$

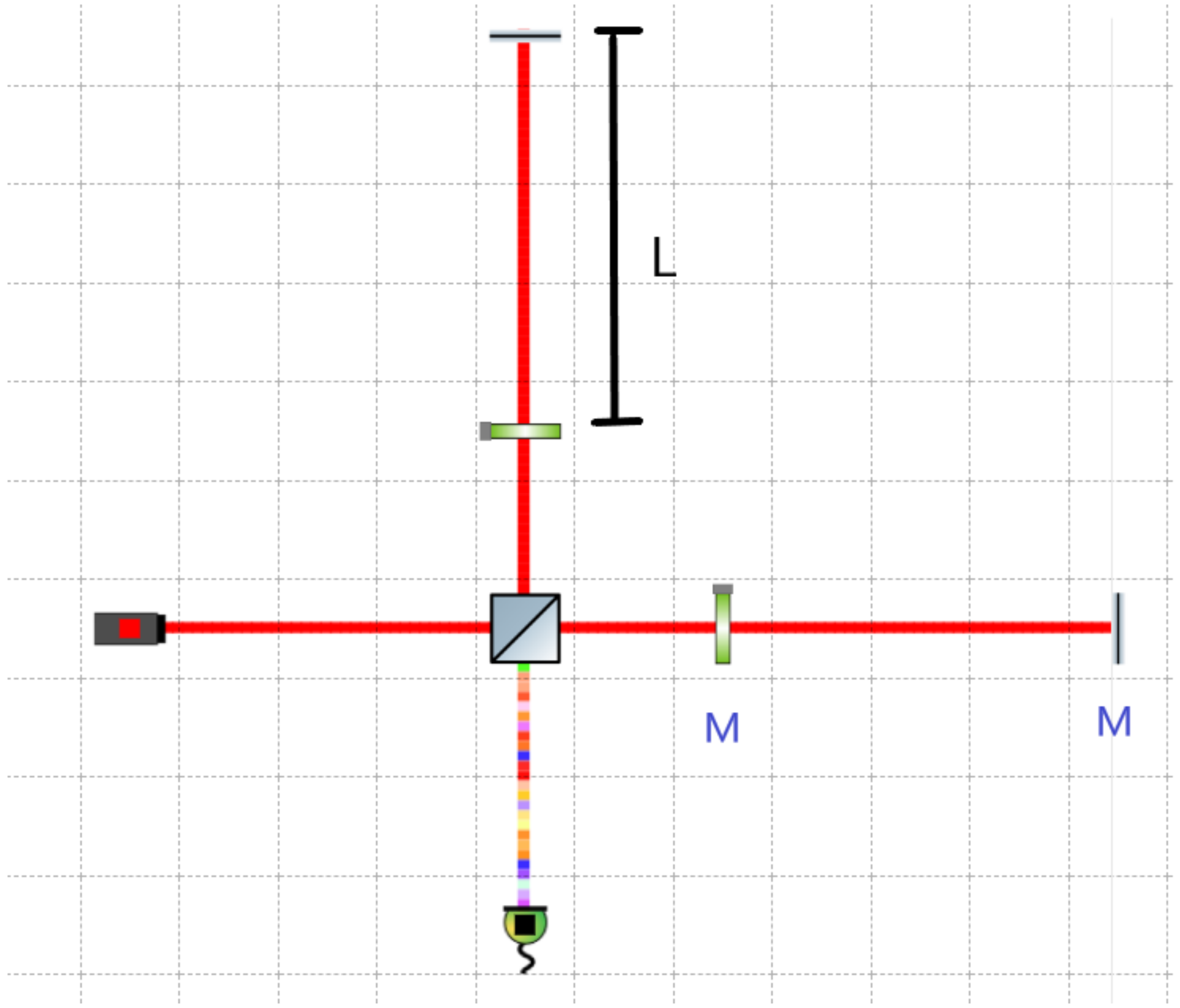


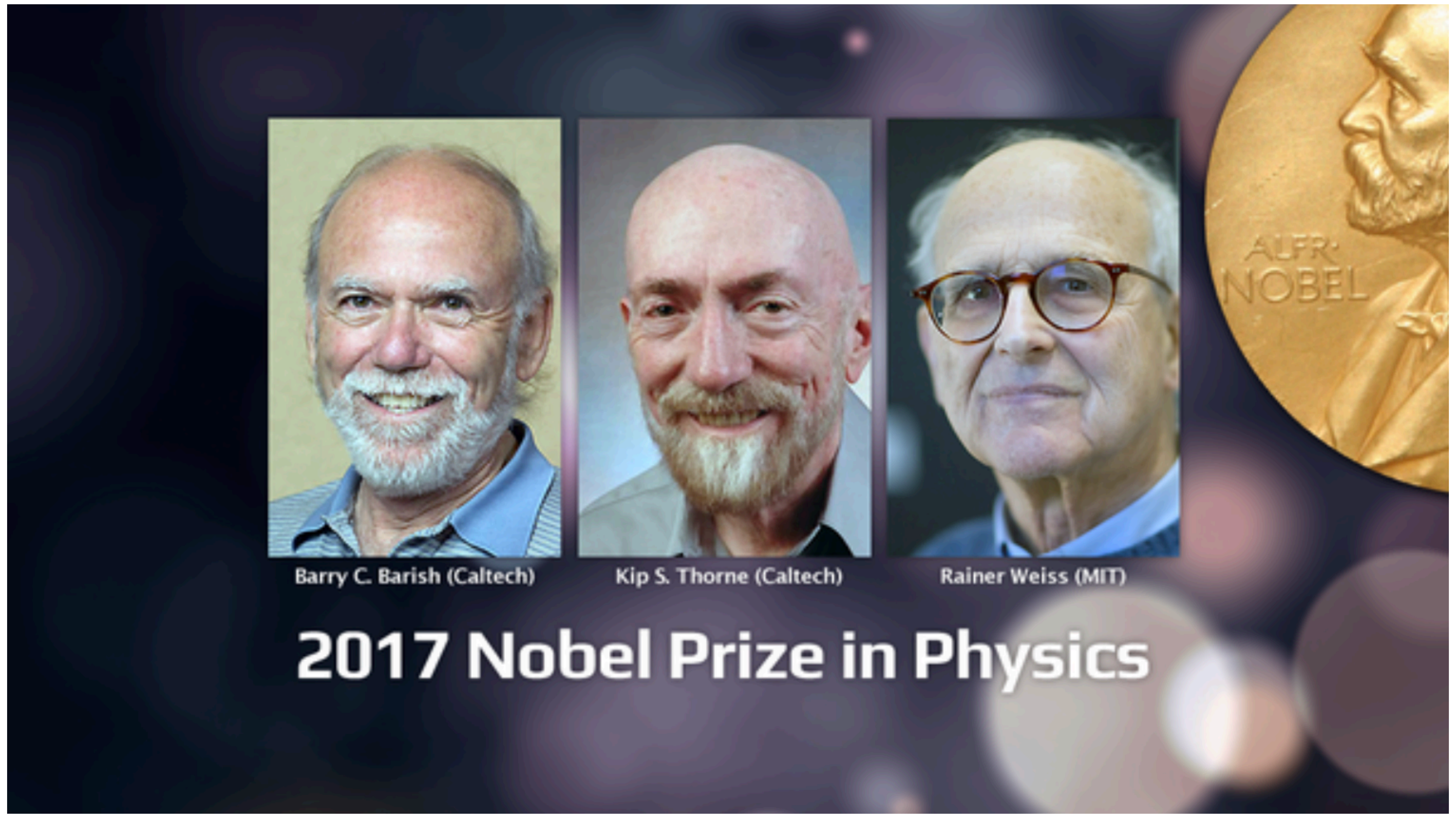


Images from: https://www.science.org/content/article/crash-titans-imminent-merger-giant-black-holes-predicted, https://www.ligo.caltech.edu/page/gw-sources

Detection

- Interferometry

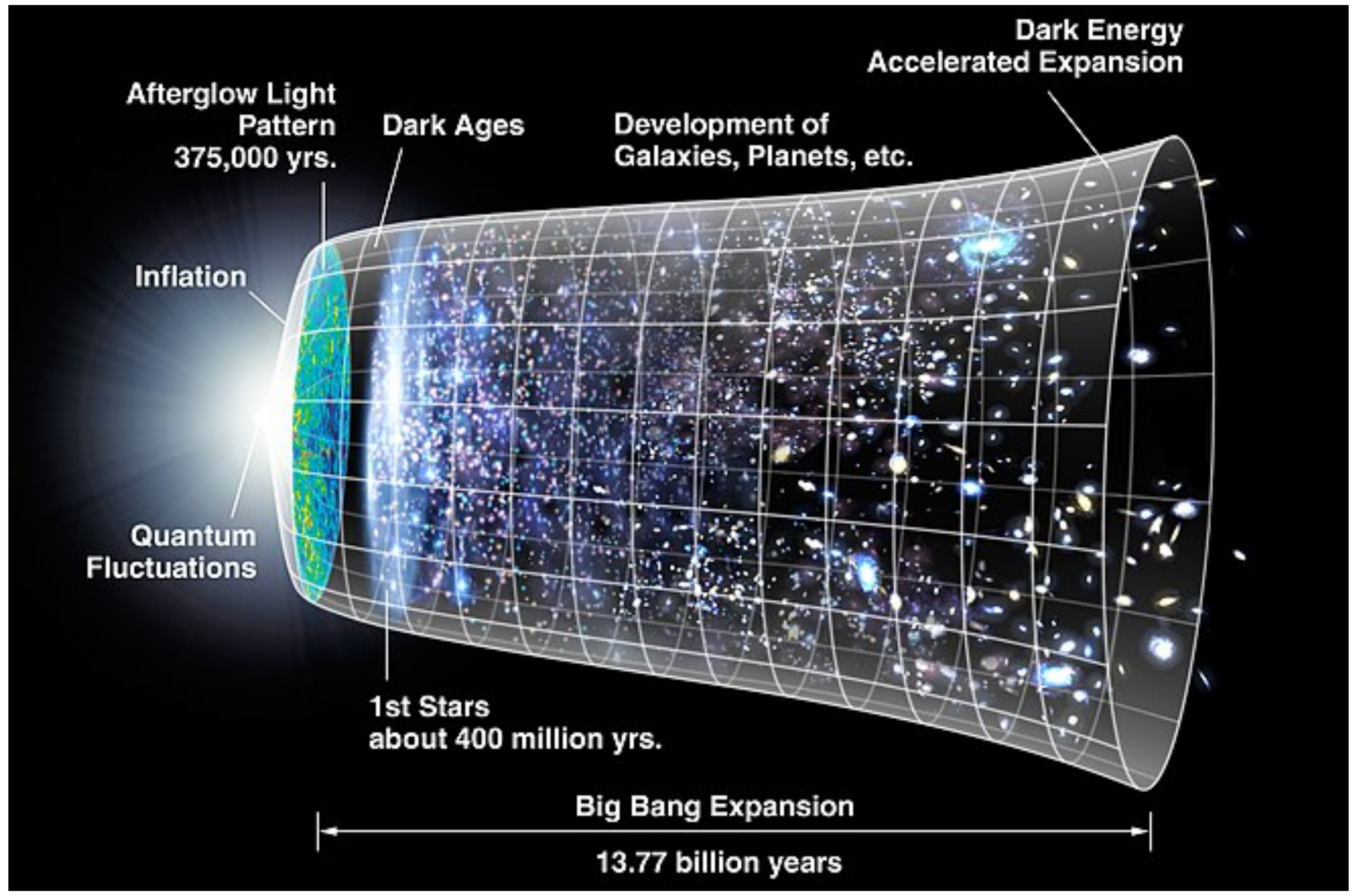




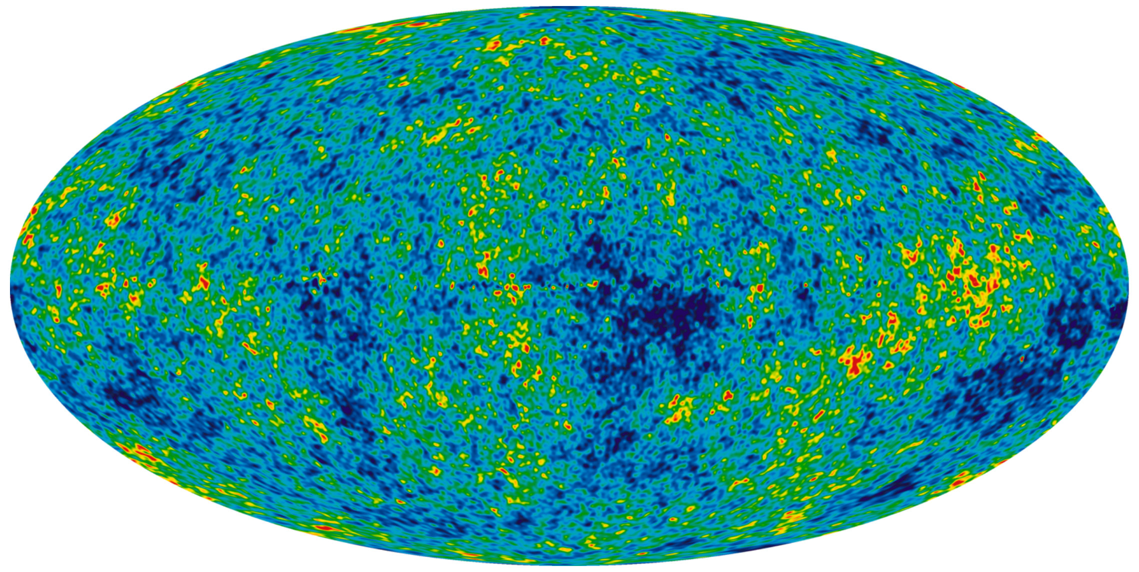
https://www.ligo.caltech.edu/page/about

Primordial Gravitational Waves

- Stochastic background from early stage of the Universe
- Inflationary GW (Gravitational Waves)?



Microwave

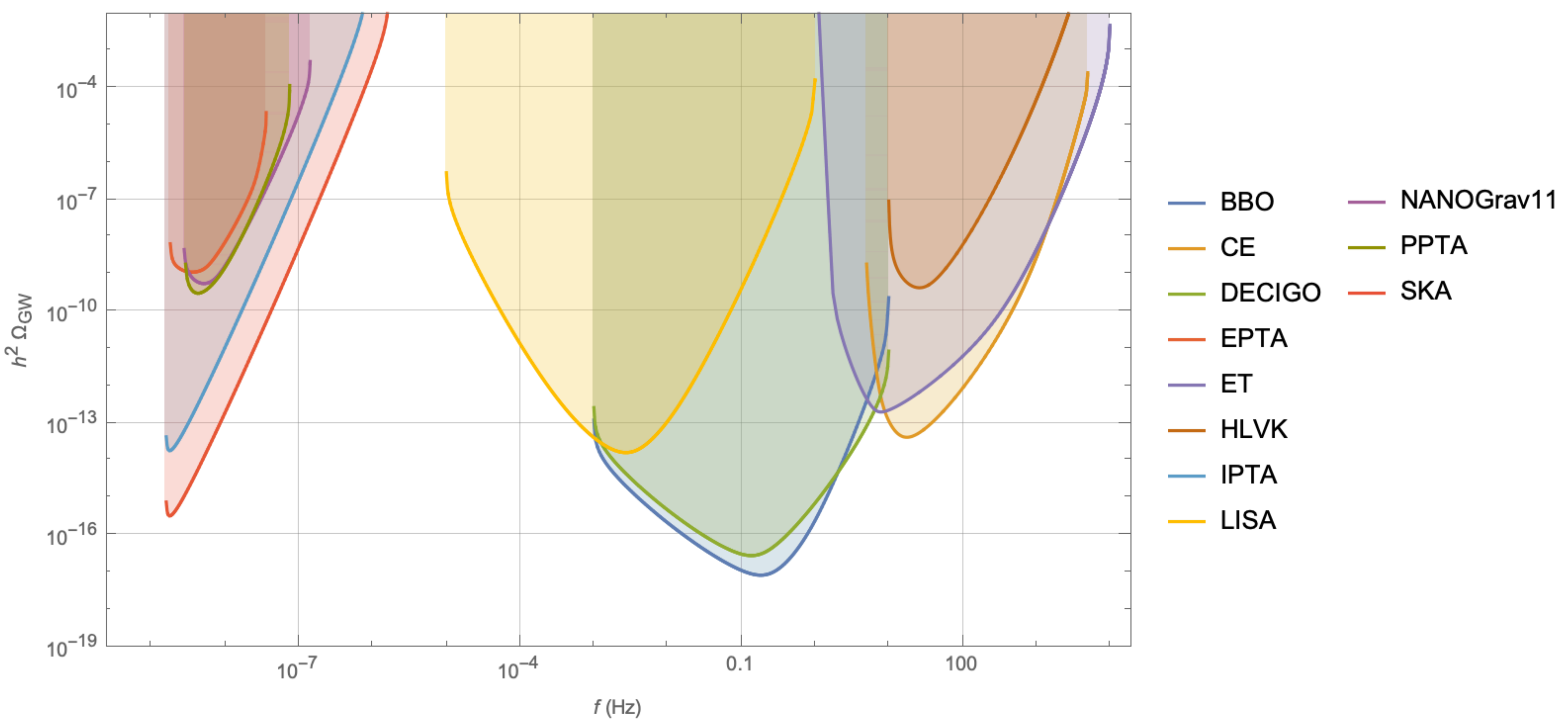


GR Wave

???

Image credit: NASA, ESA, CSA, and STScI

Detectability



But stochastic GW intensity usually falls below $\sim 10^{-17}$

Cosmological History

- Equation of State $w = \frac{p}{\rho} = \frac{T^{ii}}{T^{00}}$

- Cosmological Energy conservation Eq $\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$

- Matter Dominant (w=0)

  $\rho \propto a^{-3}$

- Radiation Dominant (w=1/3)

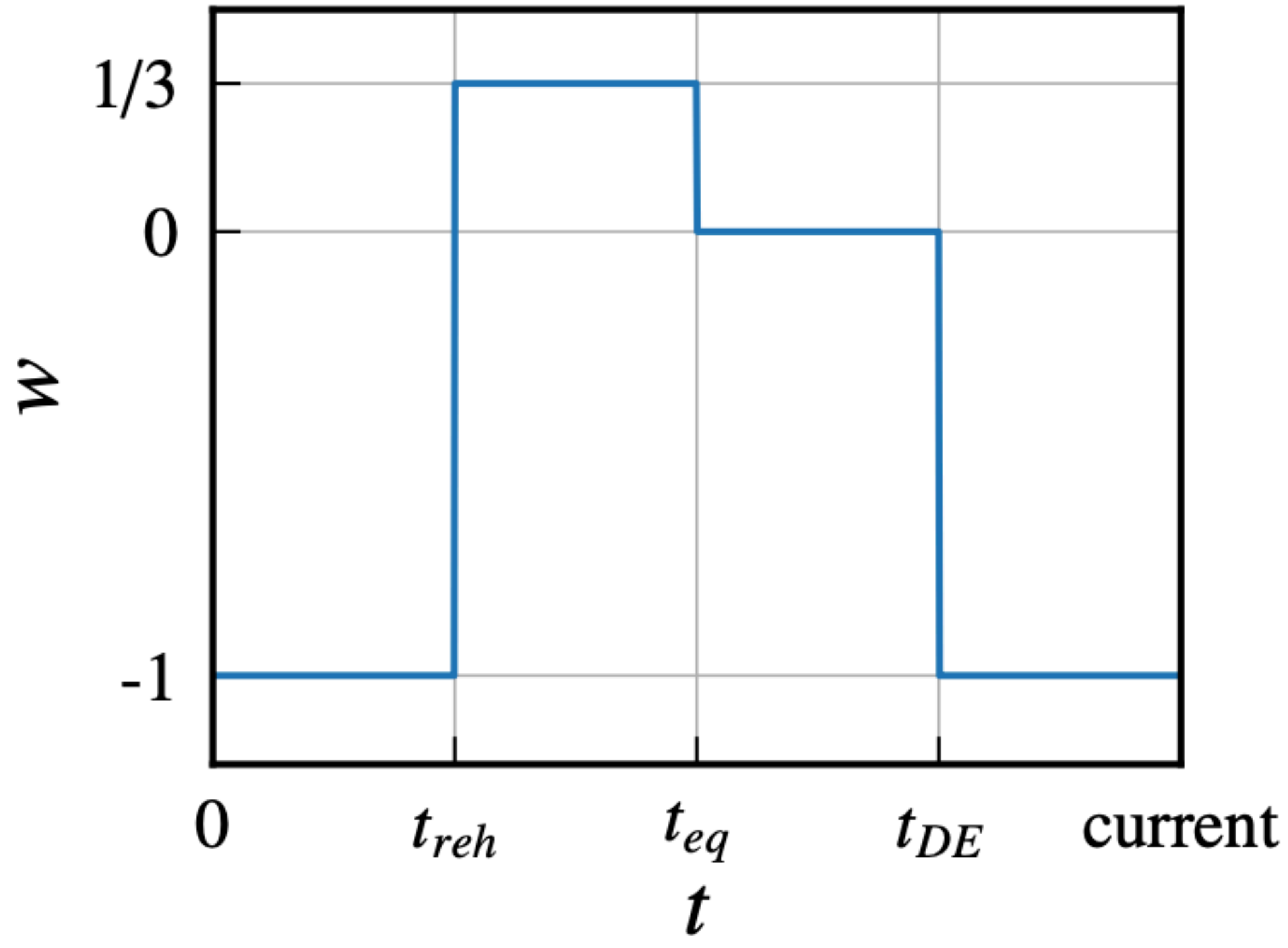
  $\rho \propto a^{-4}$

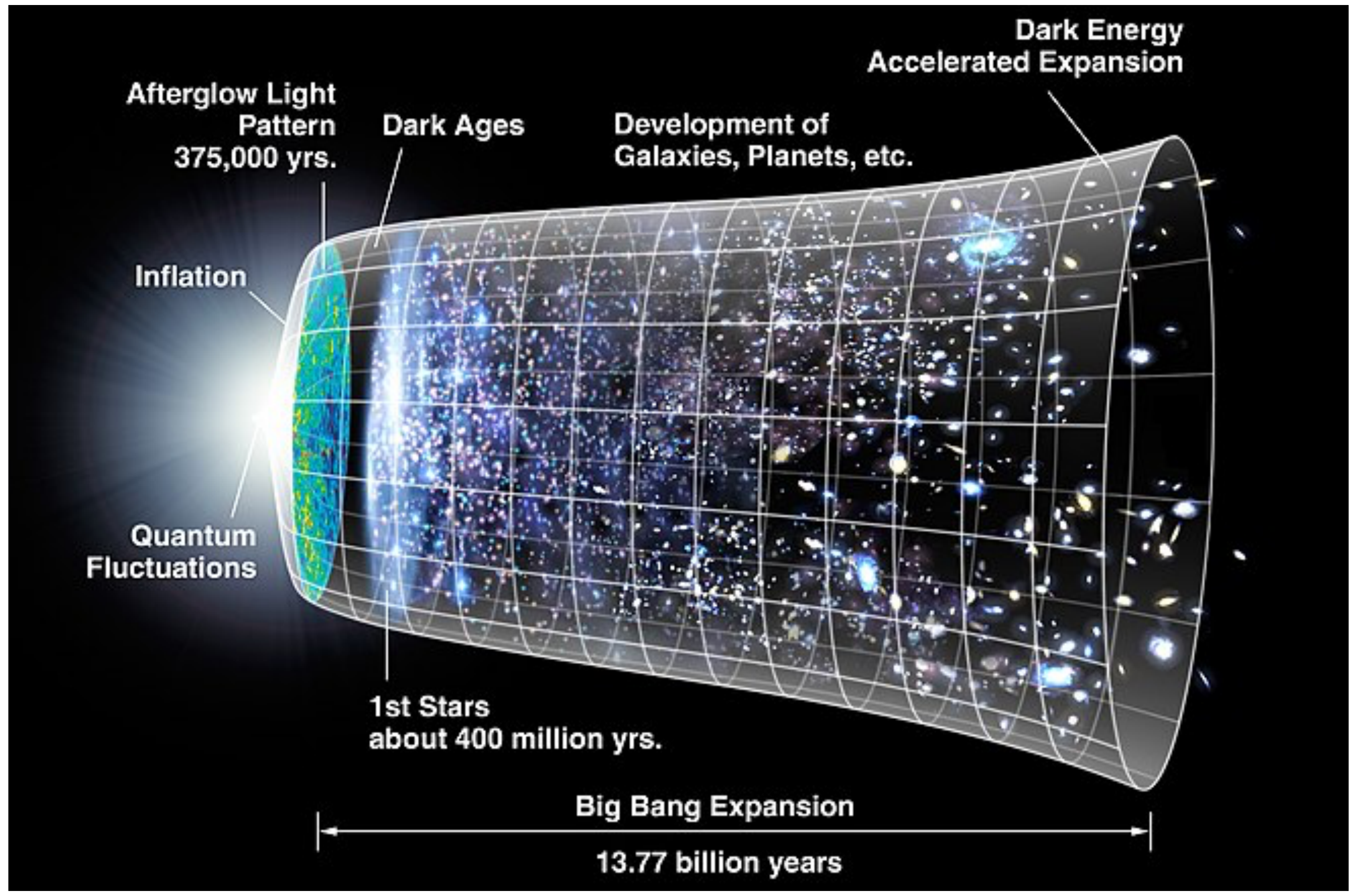
- Vacuum Dominant (w= -1)

  $\rho \propto a^{0}$

Standard Cosmology

- ΛCDM History





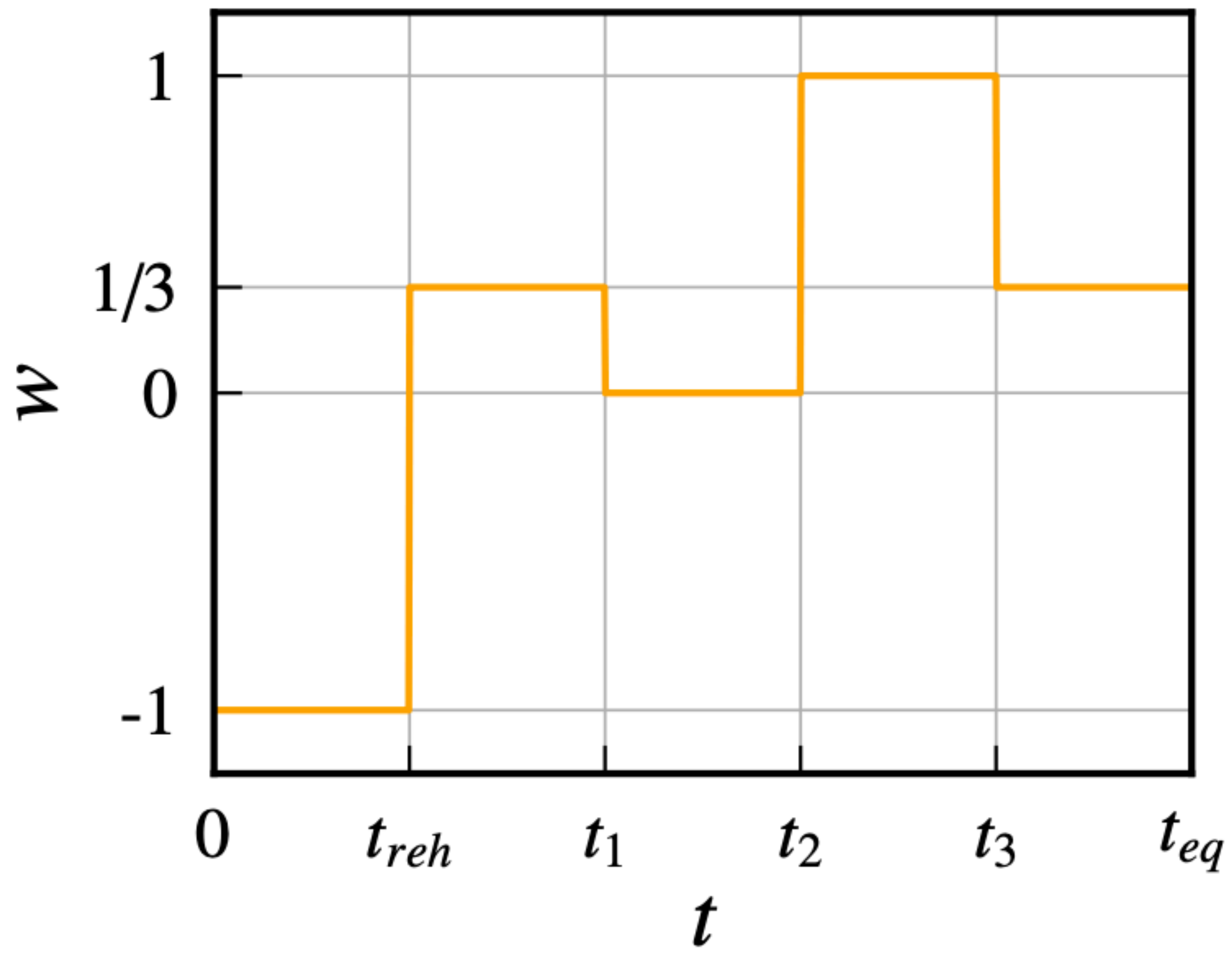
Kination (Non-standard Cosmology)

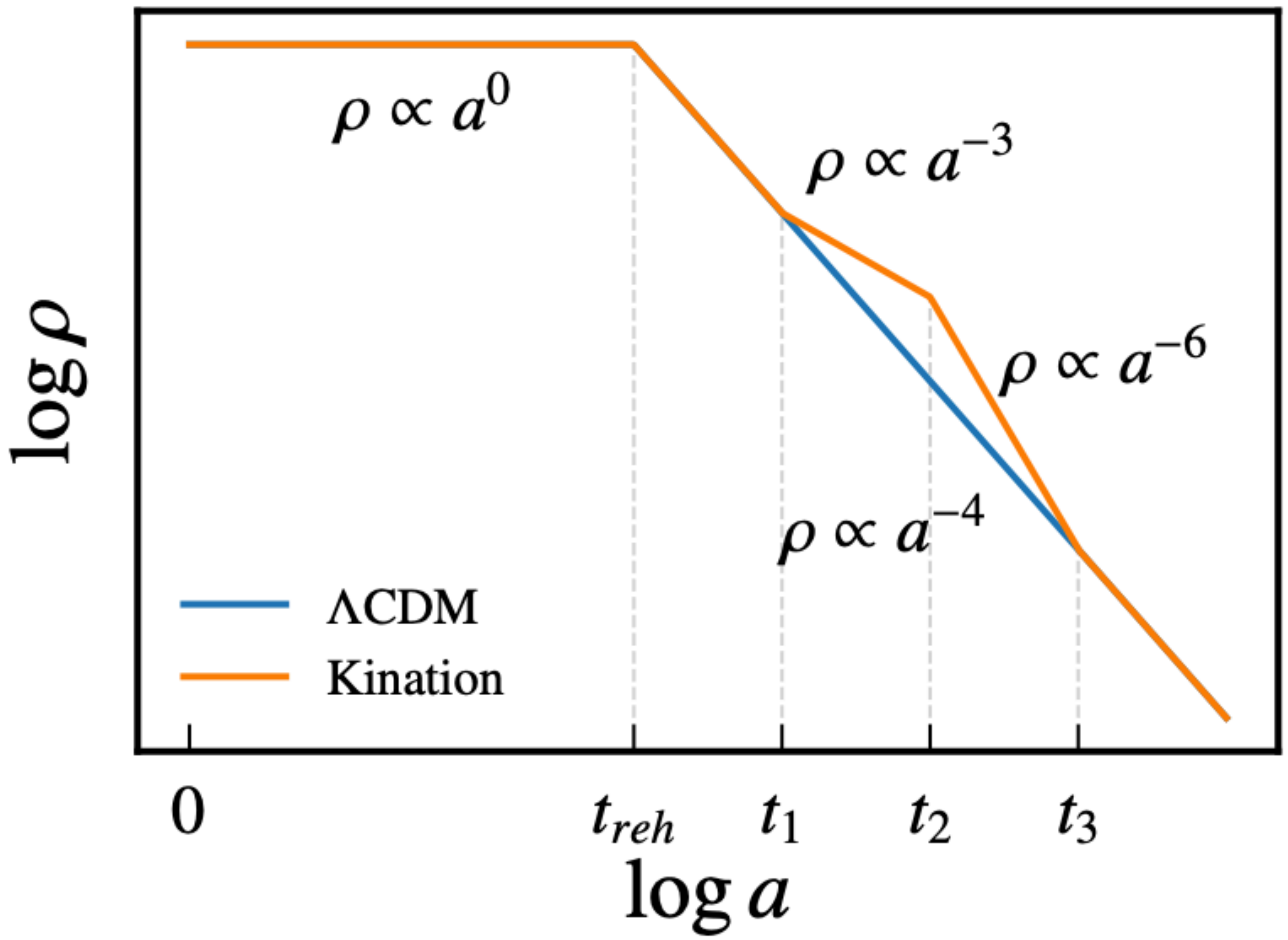
Stage in which Kinetic Energy of scalar field is dominant.

$$w = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \simeq 1$$

Predicted from many different models (e.g. Axion Kination)

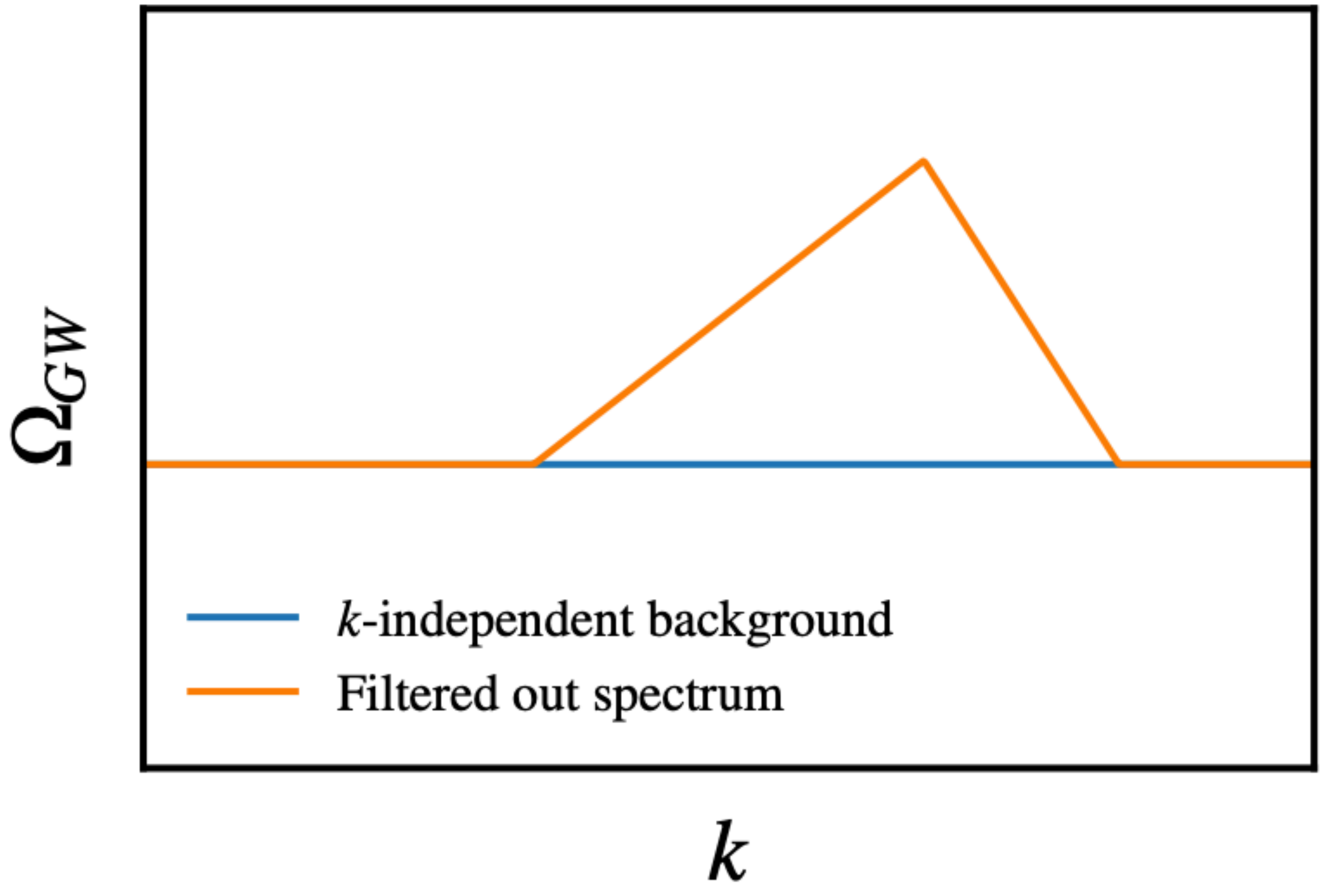
$$\rho \propto a^{-6}$$

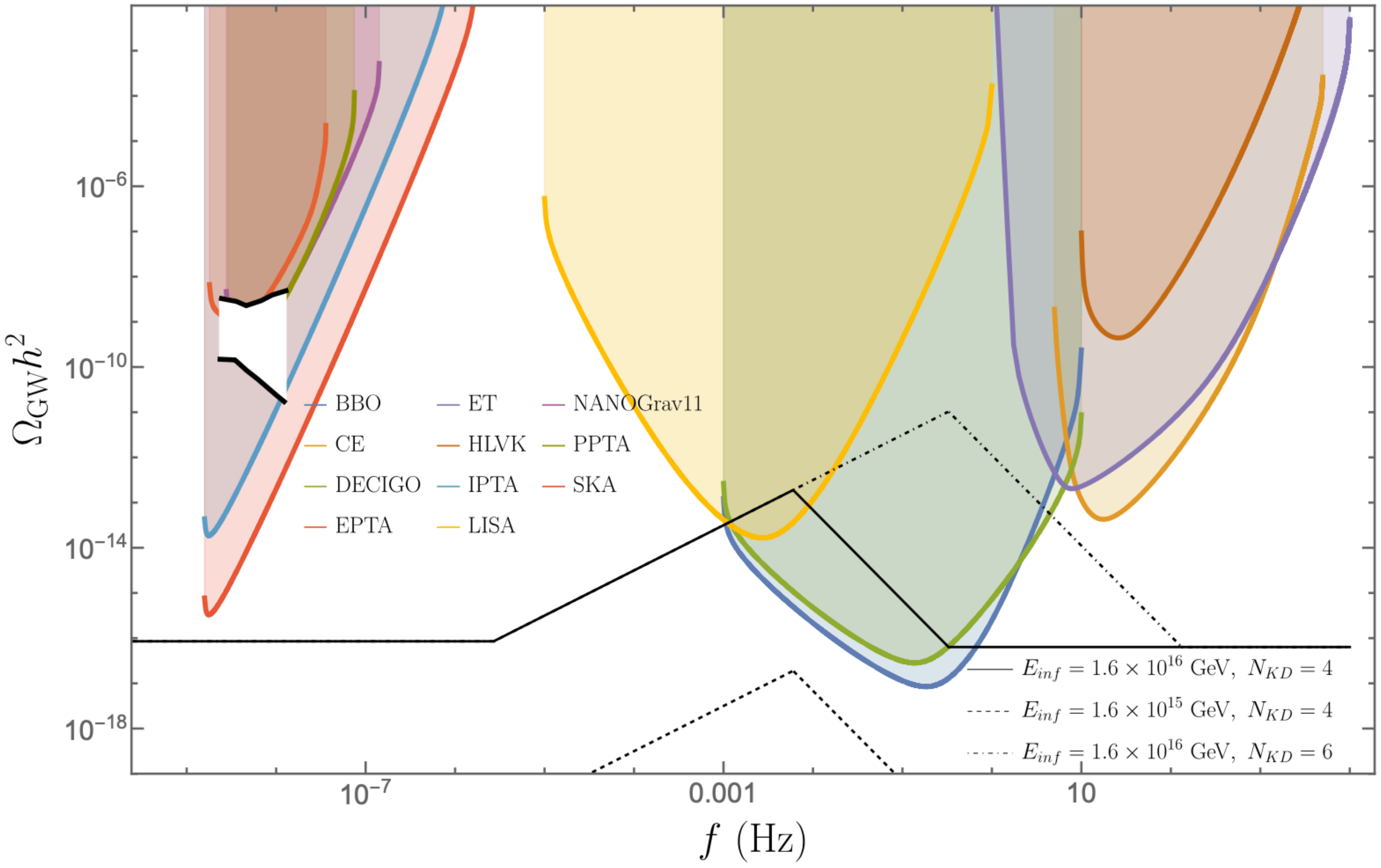




log ρ
ρ ∝ a^0
ρ ∝ a^−3
ρ ∝ a^−6
ρ ∝ a^−4
ΛCDM
Kination
0
t_reh
t_1
t_2
t_3
log a

Peak from Stochastic GW





Detectable?

Previous Studies

$$\Omega_{GW} = \frac{\rho_{GW,0}}{\rho_{tot,0}} = \left(\frac{\rho_{GW,*}}{\rho_{tot,*}}\right)\left(\frac{H_*}{H_0}\right)^2\left(\frac{a_*}{a_0}\right)^4$$

1) Friedmann eq $\rho_{tot} = 3M_{pl}^2 H^2$

2) GW are massless fields $\rho_{GW} \propto a^{-4}$

f-independent $\frac{\rho_{GW,*}}{\rho_{tot,*}}$ at the production

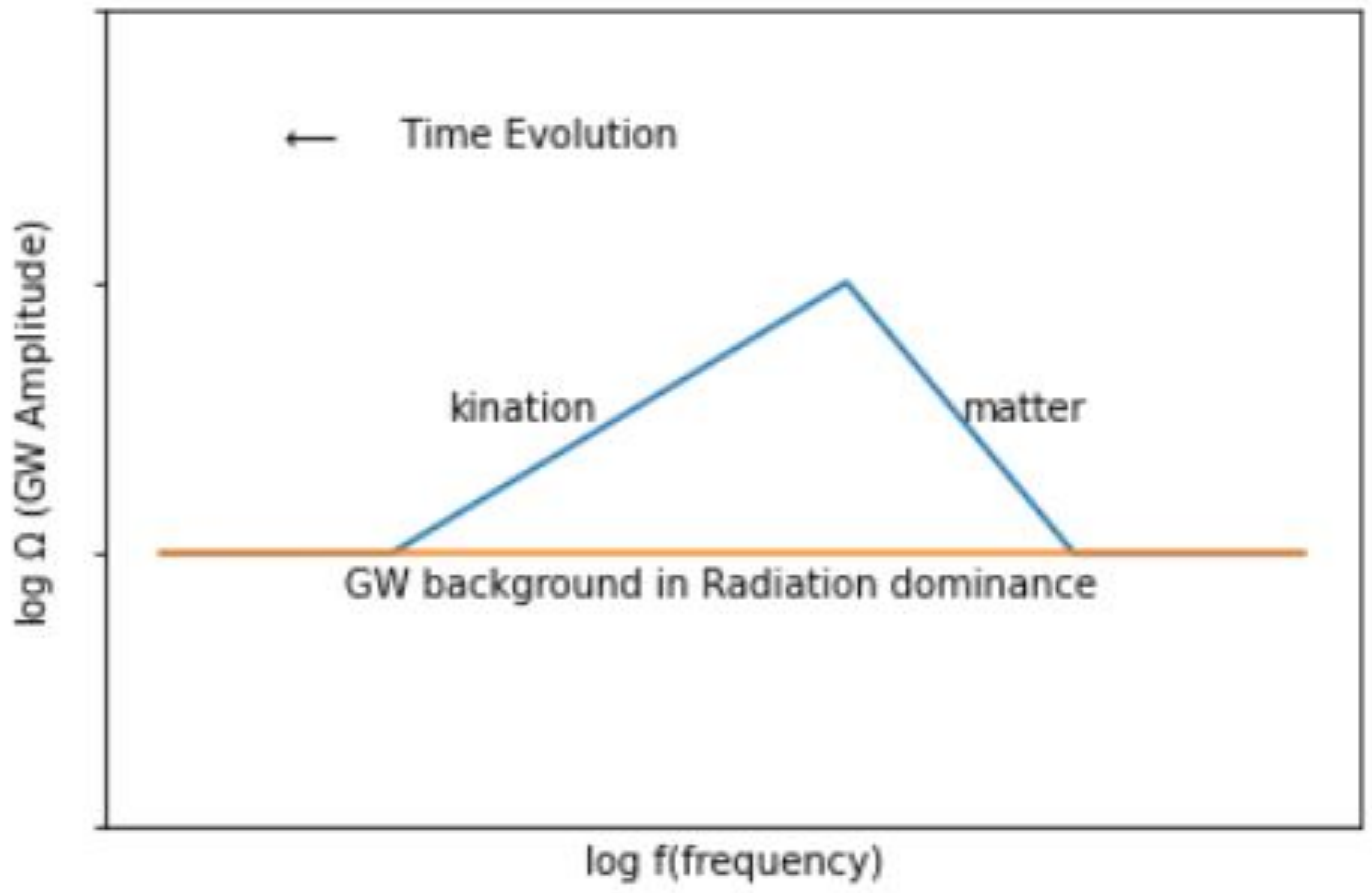
$$f \propto a_*^{-(1+3w)/2}, \quad a^2 \propto f^{-\frac{4}{1+3w}}$$

$$\Omega_{GW} \propto H_*^2 \cdot a_*^4 \propto f^\beta \quad \left(where \quad \beta \equiv -2\left(\frac{1-3w}{1+3w}\right)\right)$$

$$MD \; : \; w = 0, \quad \beta = -2 \rightarrow \quad \Omega_{GW} \propto f^{-2}$$

$$RD \; : \; w = \frac{1}{3}, \quad \beta = 0 \rightarrow \quad \Omega_{GW} \propto f^0$$

$$KD \; : \; w = 1, \quad \beta = 1 \rightarrow \quad \Omega_{GW} \propto f$$

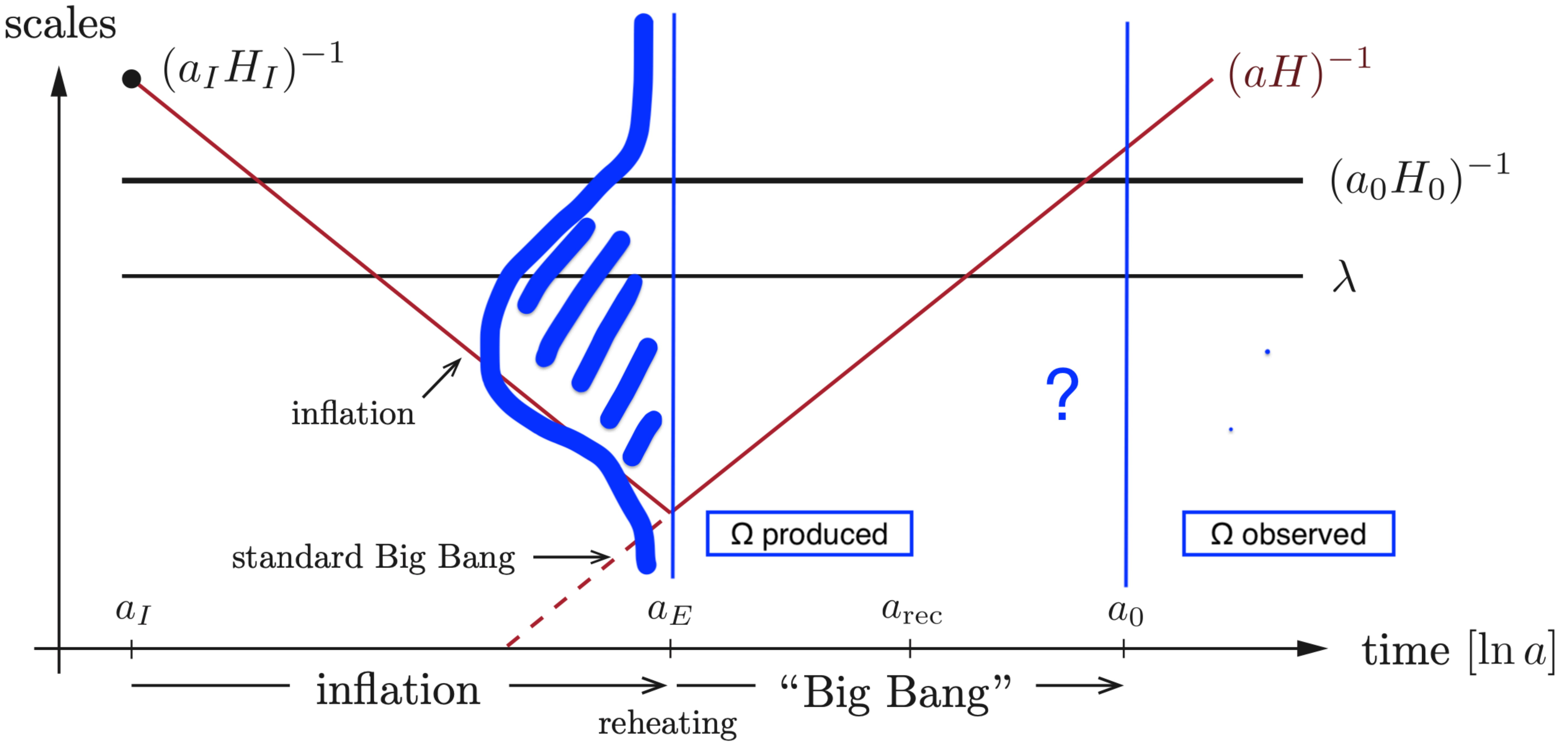


Gouttenoire, G. Servant, and P. Simakachorn, Kination cosmology from scalar fields and gravitational-wave signatures (2021)

General Mapping

- General Cosmological Histories (Other than MD-KD)
- General (not-flat) Production Spectrum at Inflationary stage

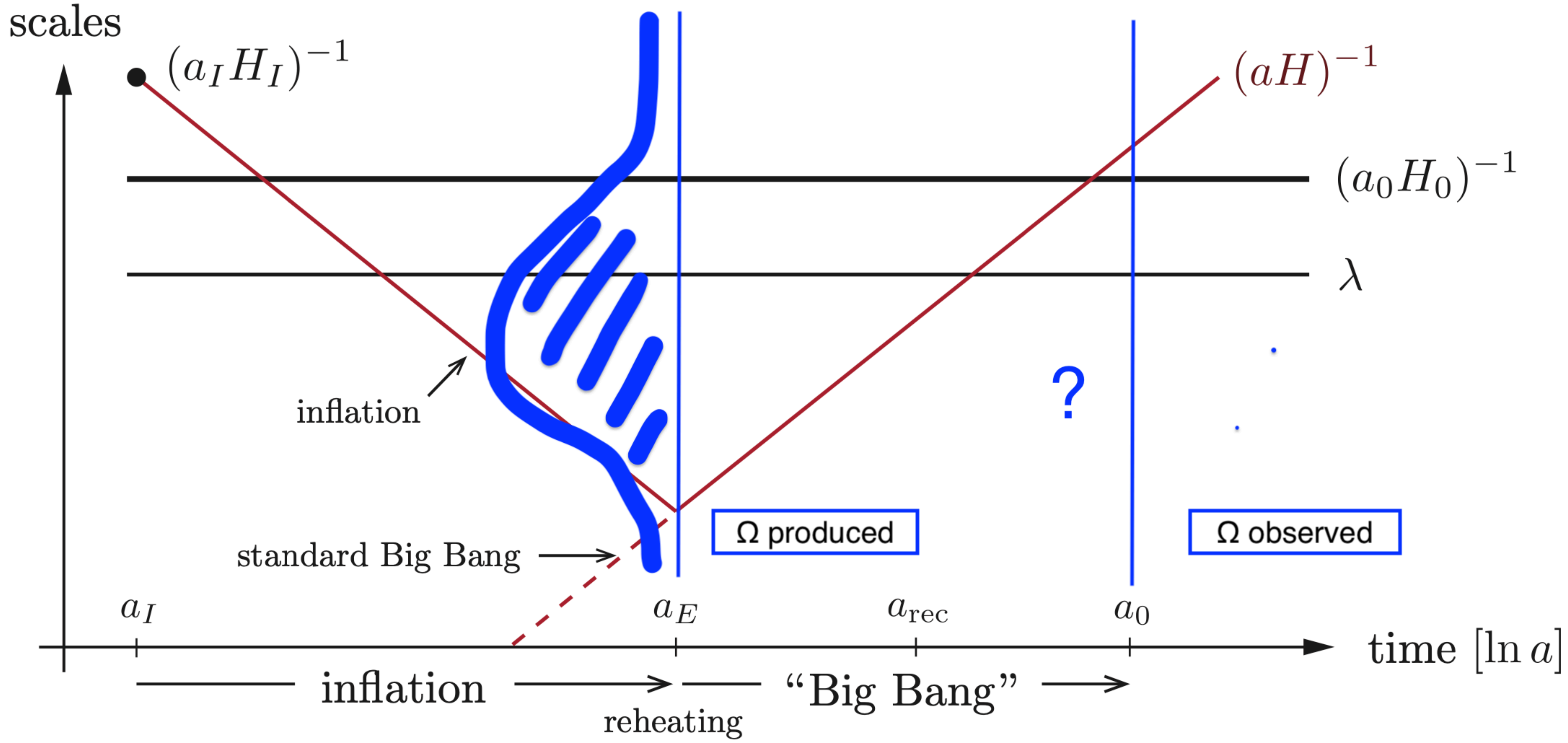
→ How will these map onto current GW spectrum?

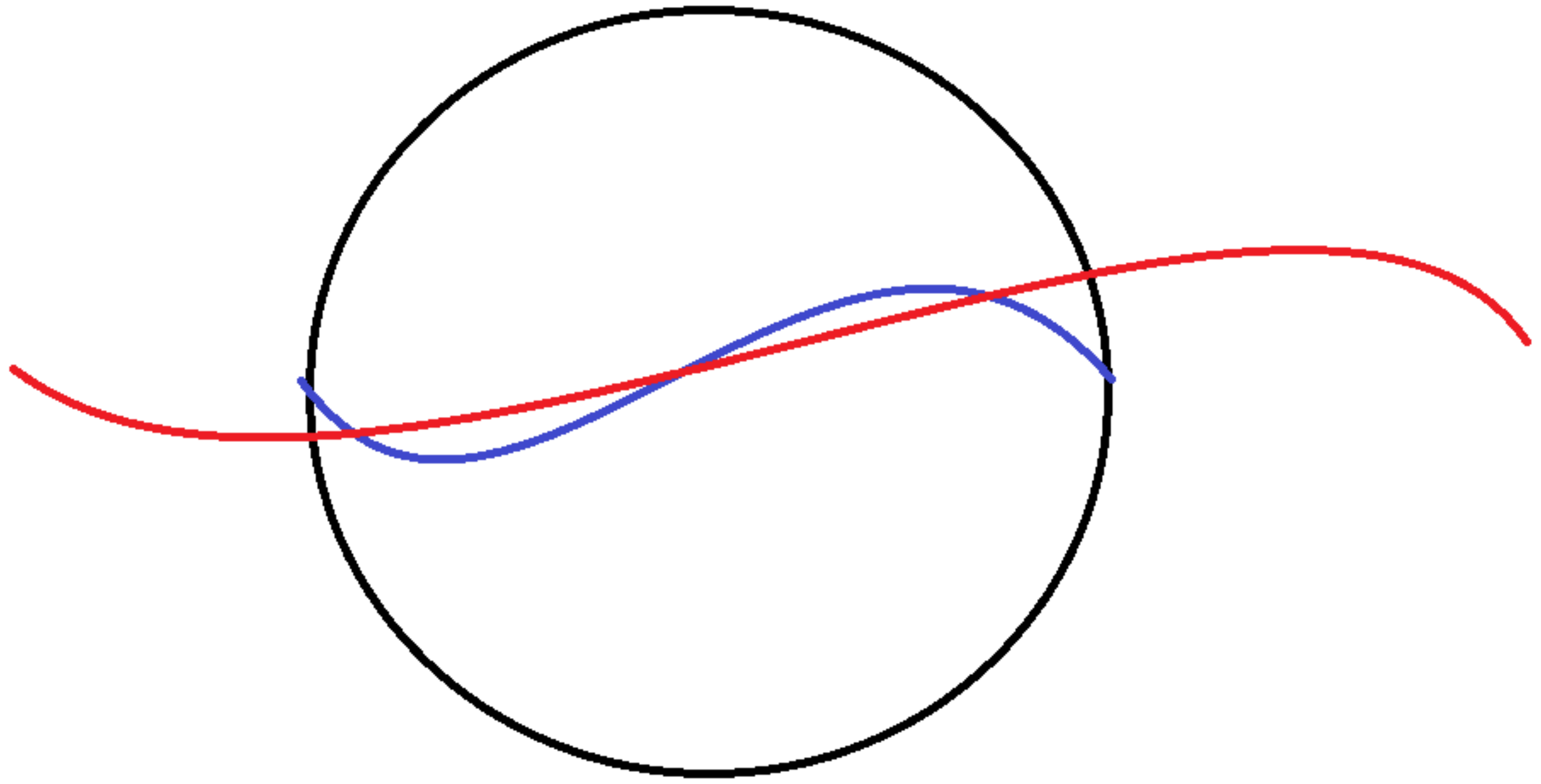


Cosmology Lecture notes, D. Baumann (2022)

Horizon Exit / Entry

- Causal contact can only be made within the comoving Hubble radius $(aH)^{-1}$
- Perturbations $h_{\mu\nu}$ in the metric tensor are frozen upon horizon exit (Inflation)
- Frozen modes continuously re-enter the horizon (standard Big-Bang evolution)





$\lambda << (aH)^{-1}$

SuperHorizon

- FRW metric

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j$$

- Einstein Equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \quad , \quad G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Linearized Equation $\quad \ddot{h}_{ij}(\vec{x}, t) + 3H\dot{h}_{ij}(\vec{x}, t) - \dfrac{\nabla^2}{a^2}h_{ij}(\vec{x}, t) = 16\pi G \Pi^{TT}_{ij}(\vec{x}, t)$

where Π^{TT}_{ij} is the transverse-traceless part of Energy momentum tensor

P. Simakachorn, Charting Cosmological History and New Particle Physics with Primordial Gravitational Waves, Ph.D. thesis, Staats-und Universit¨ atsbibliothek Hamburg Carl von Ossietzky (2022)

$$\ddot{h}_{ij}(\vec{x},t) + 3H\dot{h}_{ij}(\vec{x},t) - \frac{\nabla^2}{a^2}h_{ij}(\vec{x},t) = 16\pi G\Pi_{ij}^{TT}(\vec{x},t)$$

In the Fourier basis,

$$h_{ij}(\vec{x},t) = \sum_{\lambda=+,\times}\int\frac{d^3k}{(2\pi)^3}e^{k\cdot x}h^{\lambda}(k,t)\epsilon_{ij}^{\lambda}(k)$$

Solving this,

$$h_{\lambda}(k,\tau) = \begin{cases} \frac{A_{\lambda}}{a(\tau)}e^{ik\tau} + \frac{B_{\lambda}(k)}{a(\tau)}e^{-ik\tau}, & \text{for k>>aH (sub-Horizon) ,} \\ A_{\lambda}(k) + B_{\lambda}(k)\int^{\tau}\frac{d\tau'}{a^2(\tau')}, & \text{for k<<aH (super-Horizon)} \end{cases}$$

$$k = \frac{2\pi}{\lambda}$$

where $d\tau = \frac{dt}{a}$ is the conformal time

$A_{\lambda}(k)$ frozen in the Superhorizon!

$\Omega_{GW,*} = \rho_{GW}/\rho_{tot}$ at Horizon Re-Entry ?

- Spatial Temporal average by the Ergodic Theorem[1]

$$\left\langle h_\lambda(k,\tau)h_{\lambda'}(k',\tau')\right\rangle = \frac{8\pi^5}{k^3}h_c^2(k,\tau)\delta^{(3)}(k-k')\delta(\tau-\tau')\delta_{\lambda\lambda'}$$

and $\left\langle h_\lambda(x,\tau)h_\lambda(x,\tau)\right\rangle = 2\int d(logk)h_c^2(k,\tau)$

- The energy density of GW at re-entry is 00- of the energy momentum tensor :

$$\rho_{GW} = \frac{\left\langle h'_{ij}(x,\tau)h'_{ij}(x,\tau)\right\rangle}{32\pi G a^2}$$ where $'$ is the derivative w.r.t conformal time.

[1]Chiara Caprini and Daniel G Figueroa 2018 *Class. Quantum Grav.* **35** 163001
P. Simakachorn, Charting Cosmological History and New Particle Physics with Primordial Gravitational Waves, Ph.D. thesis, Staats-und Universit¨ atsbibliothek Hamburg Carl von Ossietzky (2022)

$$\rho_{GW} = \frac{\left\langle h'_{ij}(x,\tau) h'_{ij}(x,\tau) \right\rangle}{32\pi G a^2}$$

Derivative taken on the oscillatory part $h_\lambda(k,\tau) = \frac{A_\lambda}{a(\tau)} e^{ik\tau} + \frac{B_\lambda(k)}{a(\tau)} e^{-ik\tau}$ leads to

$$h'(k,\tau) \propto \frac{ik}{a} e^{ik\tau} + e^{ik\tau}\left(-\frac{a'}{a^2} \right) \sim \frac{1}{a(\tau)} e^{ik\tau}(ik) \sim (ik) \times h(k,\tau) \ ,$$

$$\rho_{GW} \sim \frac{k^2 h(k)^2 M_{pl}^2}{a^2}$$

At the moment of entry, $k = \frac{2\pi}{\lambda_{Hor}} \sim (aH)$

and therefore $\rho_{GW} \sim H^2 h^2(k) M_{pl}^2$

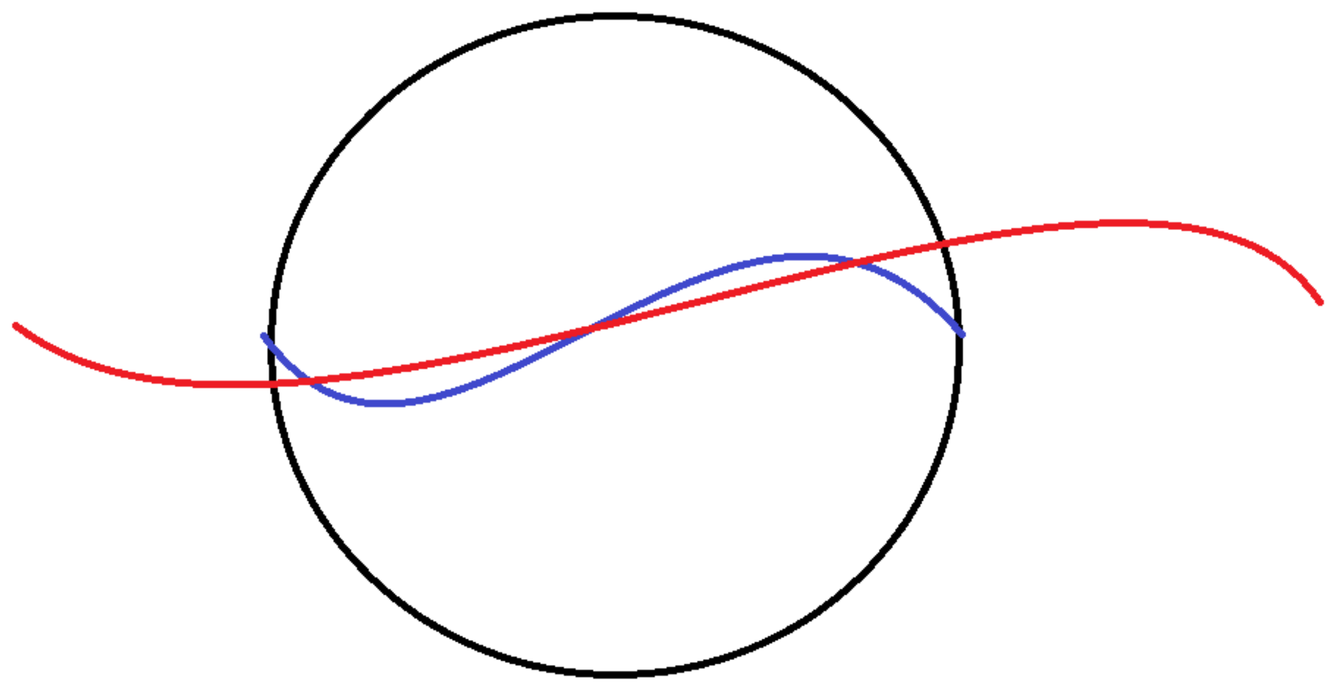
$$\rho_{GW} \sim H^2 h^2(k) M_{pl}^2$$

From the Friedmann equation $\rho_{tot} = 3M_{pl}^2 H^2$,

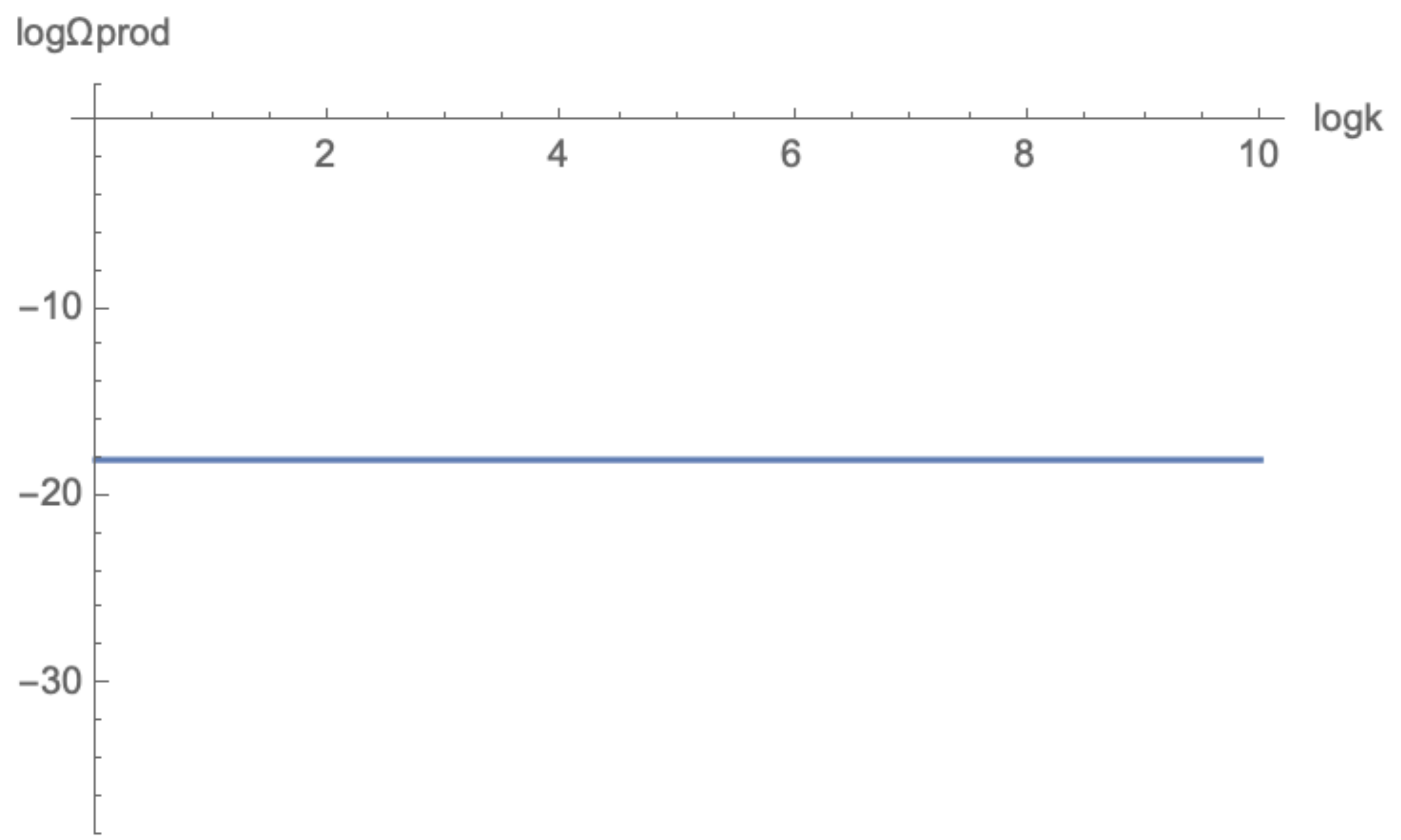
We have the energy fraction of GW at the horizon re-entry * as:

$$\Omega_{GW,*} = \frac{\rho_{GW,*}}{\rho_{tot,*}} \sim h^2(k)$$

where $h(k)$ is the metric perturbation generated mid-inflation and had been frozen in the superhorizon before the horizon re-entry.



Connection to Current Spectrum



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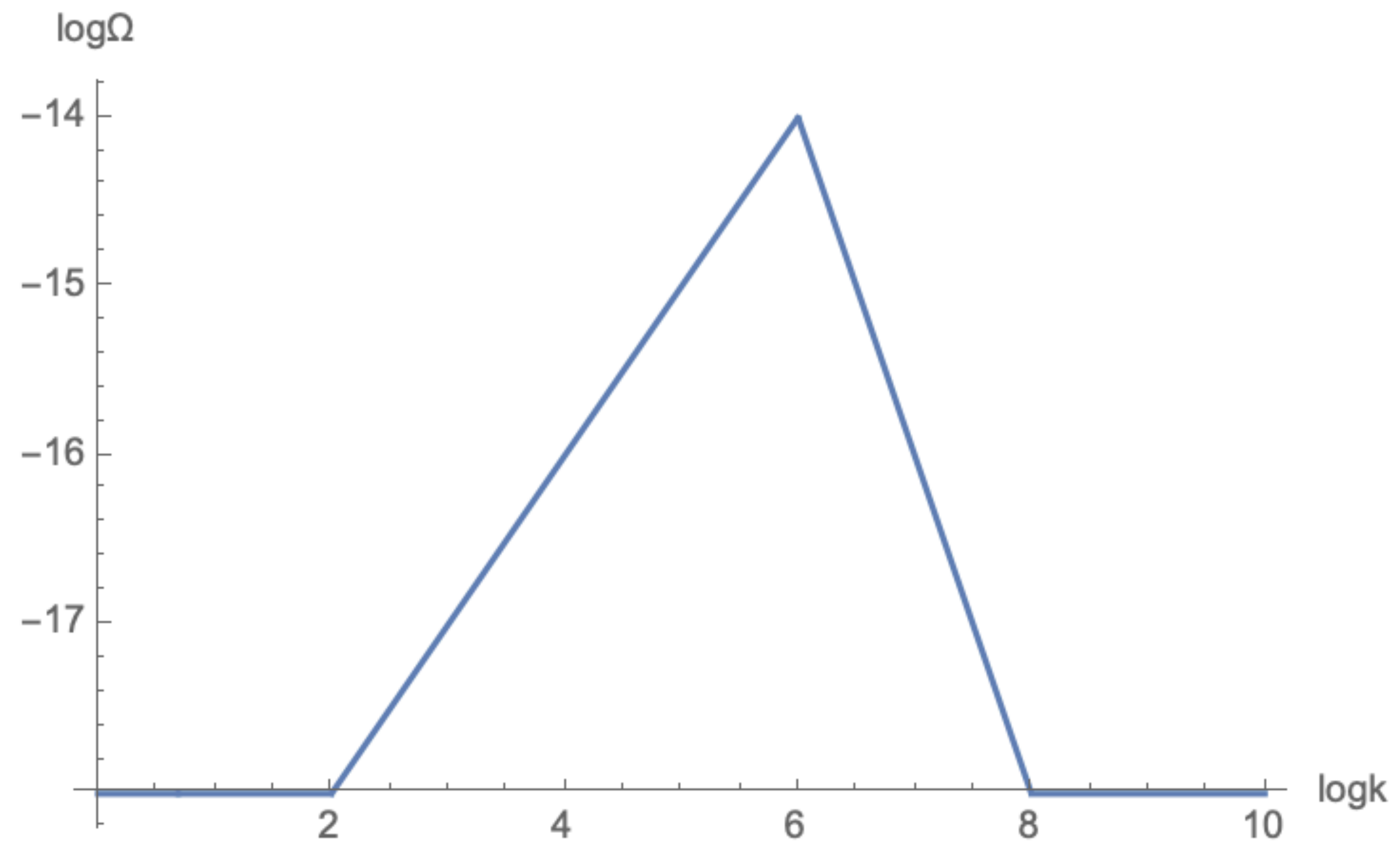
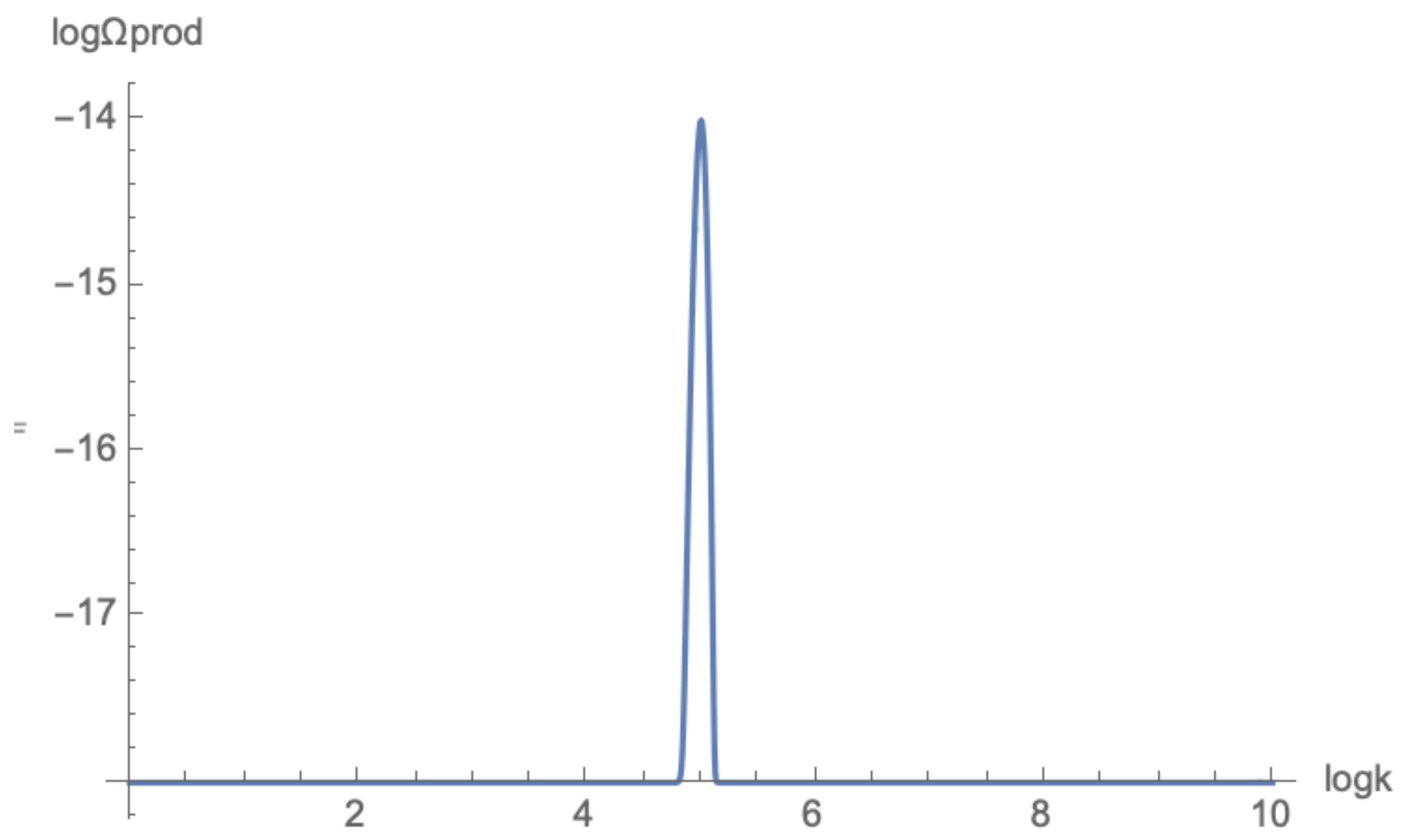




→

???

Sub Horizon?

Time Evolution of $\Omega_{GW}(k) = \frac{\rho_{GW}}{\rho_{tot}}(k)$

- Super-horizon $\left(k < aH\right)$, Frozen
- Sub-horizon $\left(k > aH\right)$, Evolves

$\rho_{GW} \propto a^{-4}, \quad \rho_{tot} \propto a^{-3(1+w)}$ at a given moment with EOS w

$\Omega_{GW}(k) = \frac{\rho_{GW}}{\rho_{tot}}(k) \propto a^{-1+3w}$ for $k > aH$

- Evolution of comoving Horizon itself

$$(aH)^{-1} \propto a^{-1}\rho^{-1/2} \propto a^{-1} \cdot a^{3(1+w)/2} = a^{\frac{1+3w}{2}}$$

In log scale,

$$log(aH)^{-1} \propto \frac{1+3w}{2} loga$$

$$log\Omega \propto (-1+3w) loga$$

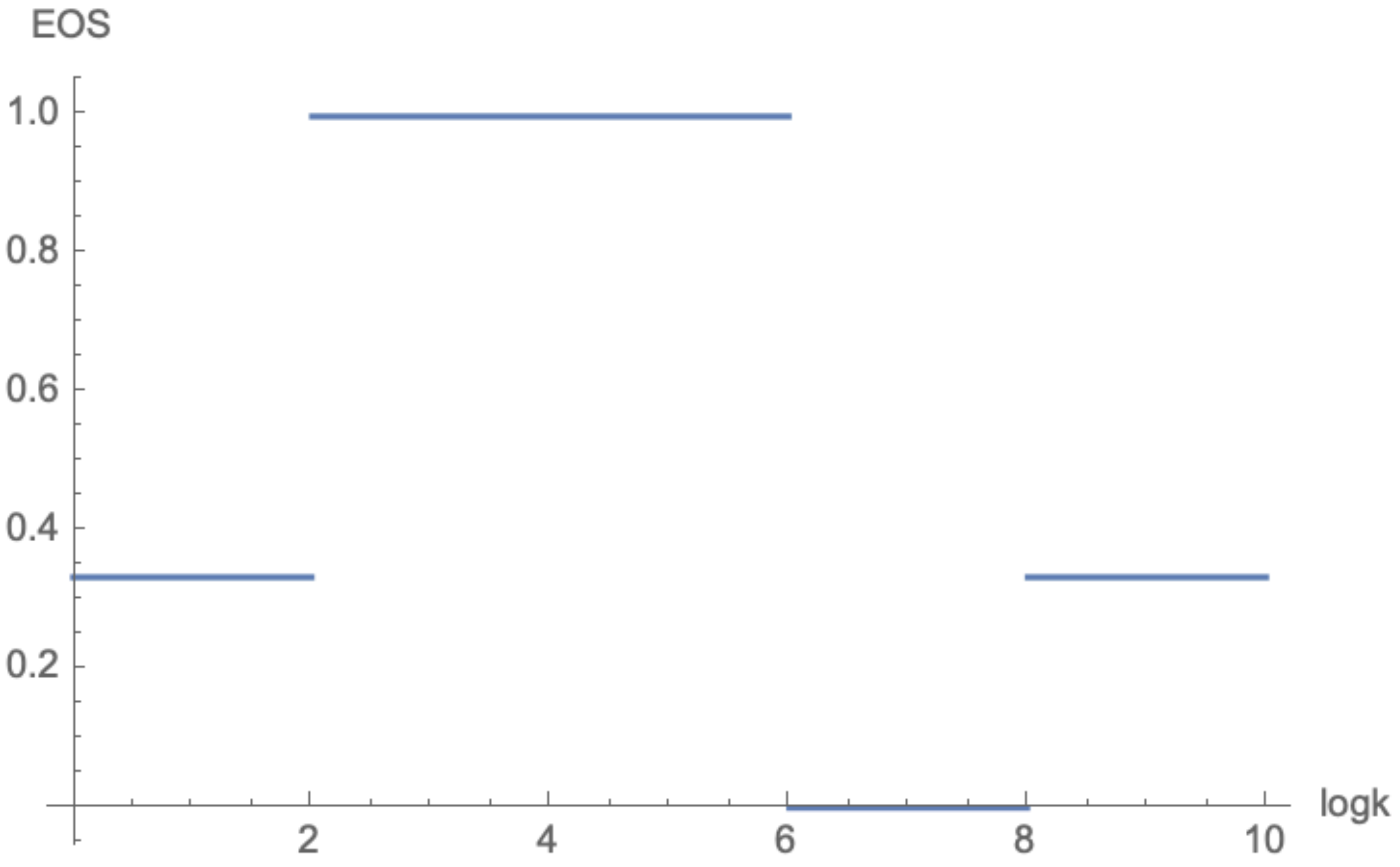
and therefore. $$\frac{dlog\Omega}{dlog(aH)^{-1}} = 2\left(\frac{-1+3w}{1+3w}\right) \equiv \beta$$

for modes inside the horizon, i.e, $k > aH$

Every time comoving horizon size $(aH)^{-1}$ changes by an order of magnitude,

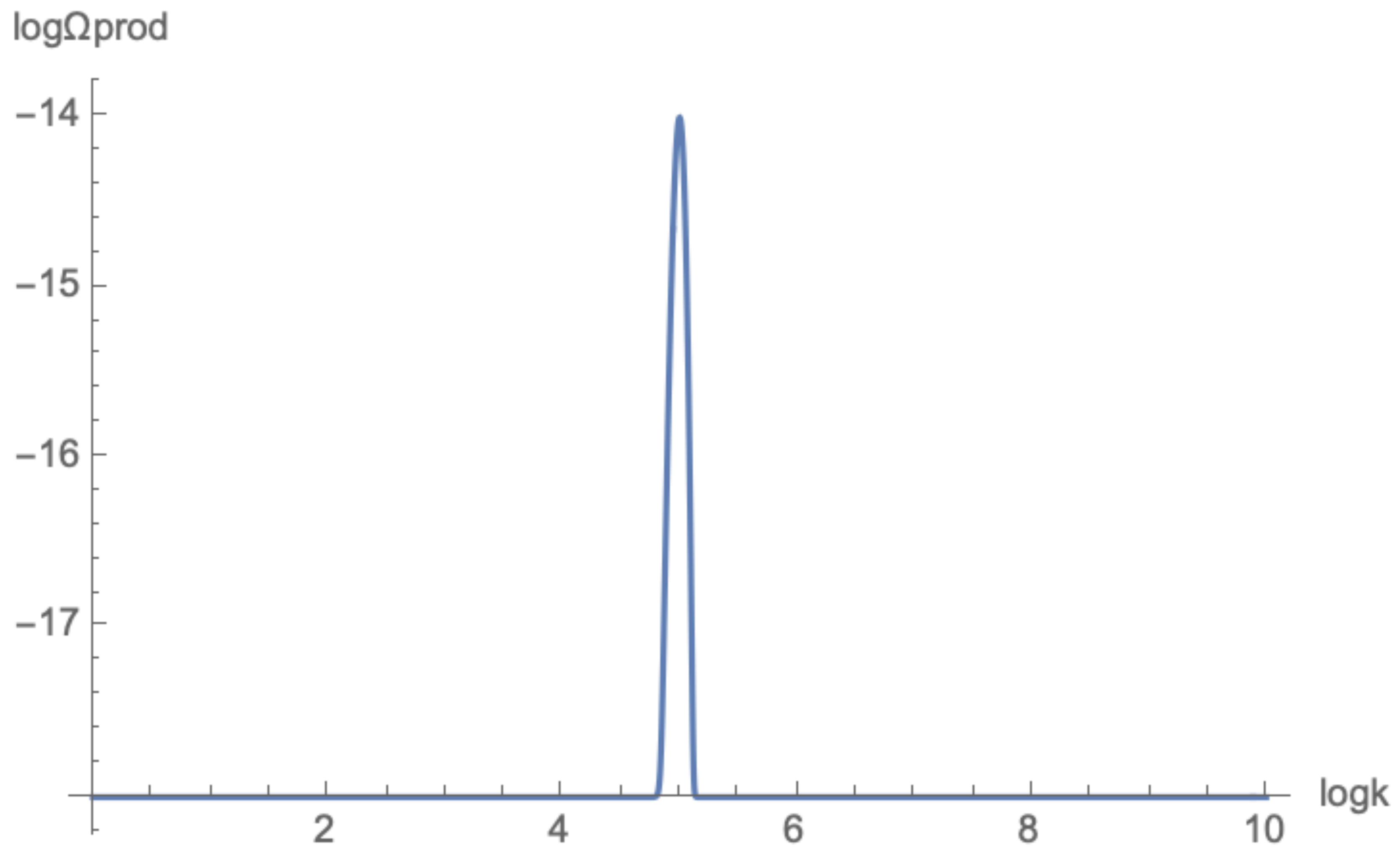
GW energy fraction $log\Omega$ for every modes $k > aH$ changes by β

Trial



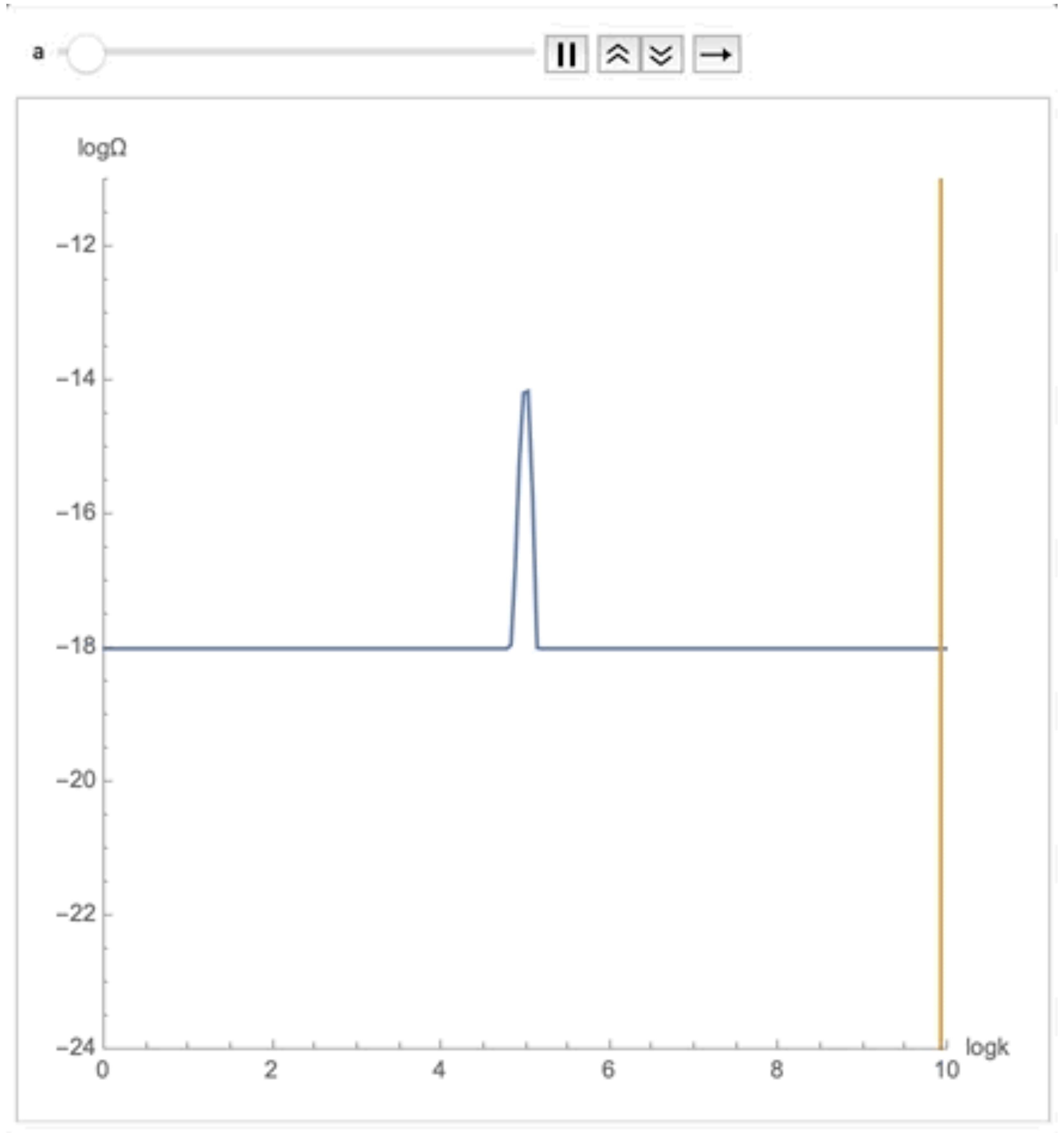
<Cosmological History>

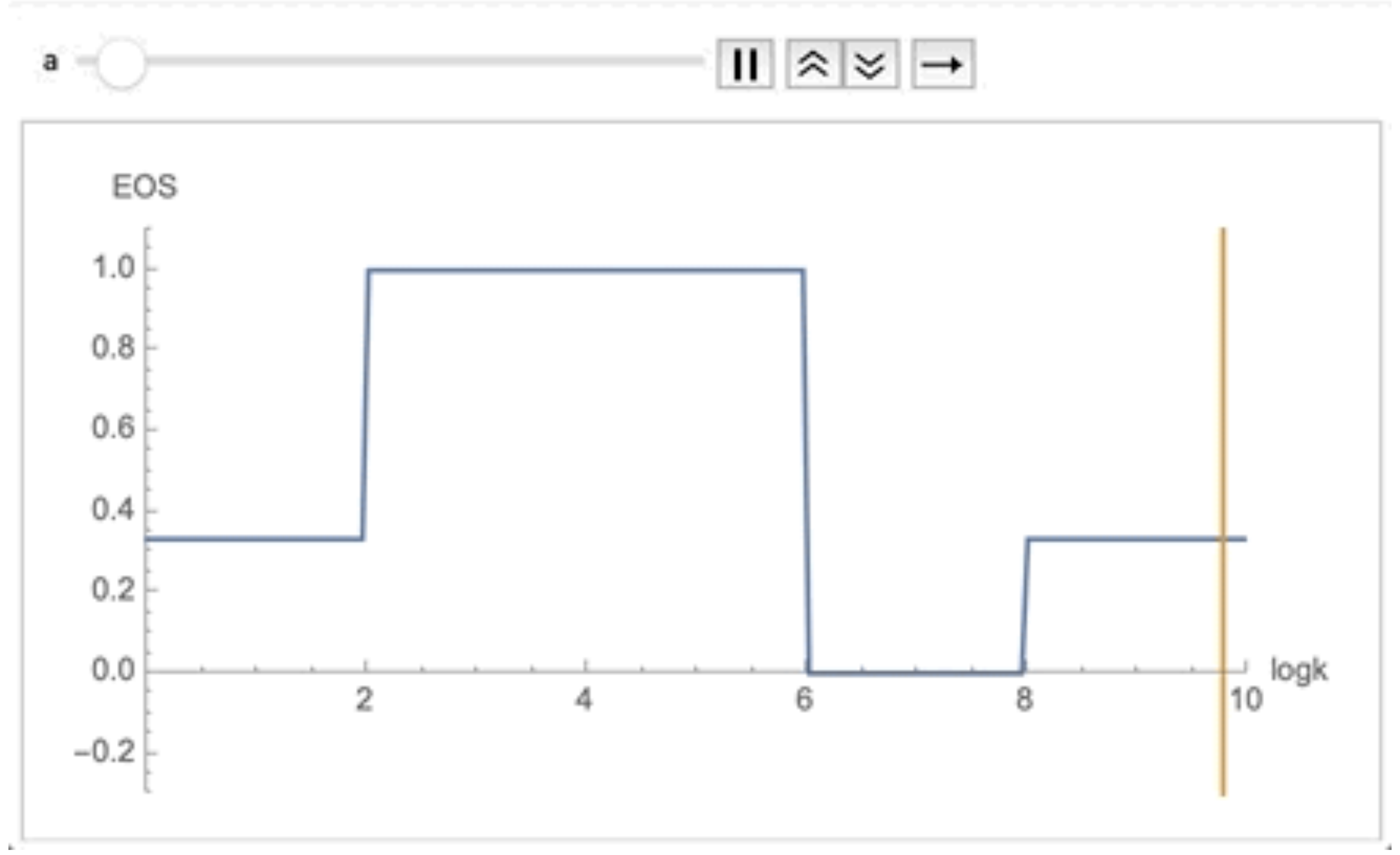
```
init = Log[10, (10 ^ (-18) * (1 + 10 000 * Exp[- ((10 ^ x - 100 000) / 10 000) ^ 2]))];
       로그                                   지수 함수
Plot[init, {x, 0, 10}, AxesLabel → {logk, logΩprod}]
플롯                    축 레이블
```



<Gaussian Spectrum at the End of Inflation>

Animation


a
logΩ
-12
-14
-16
-18
-20
-22
-24
0
2
4
6
8
10
logk


a
EOS
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
2
4
6
8
10
logk

Further Studies

- Realistic Horizon Crossing
- Numerically solving $\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$
- Quantitative Analysis using Specific models

Thank you!