Quantum states during inflation

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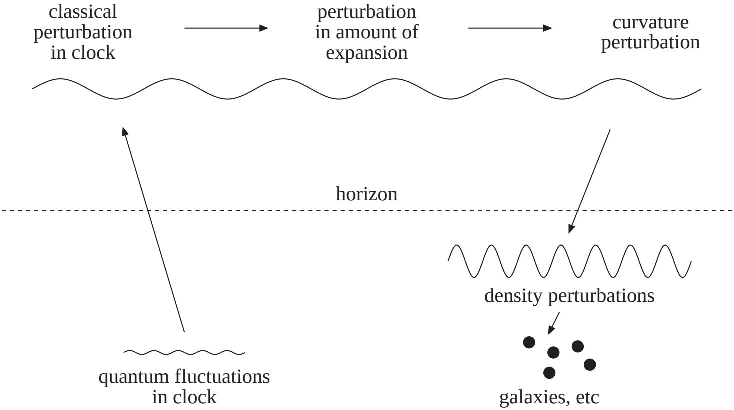
Outline

Generation and evolution of perturbations



Everything seems to be clearly understood

Perturbations and the Heisenberg description

We are interested in the perturbation variables

- Write time-dependent evolution equation
- Promote to quantum operators
- Evaluate their correlators at a certain time

Heisenberg picture for cosmological perturbations

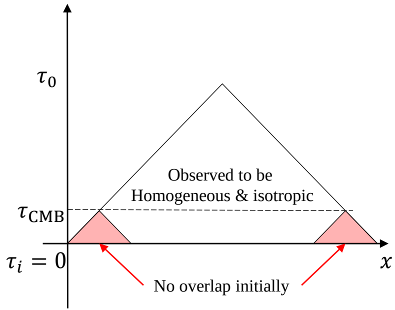
Perturbations and the Schrödinger picture?

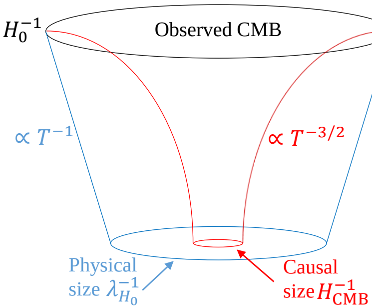
We may well give all the time dependence to the states

- Equal description
- Various choices for initial conditions
- Further applications

Observed cosmic microwave background

Troubles with CMB



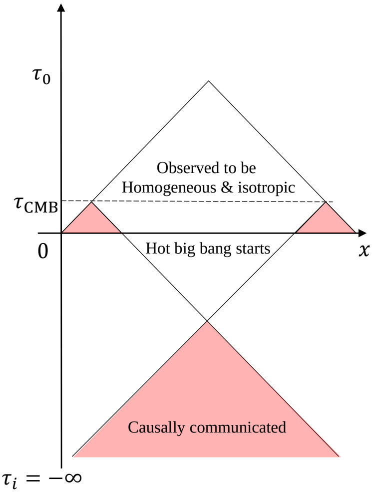


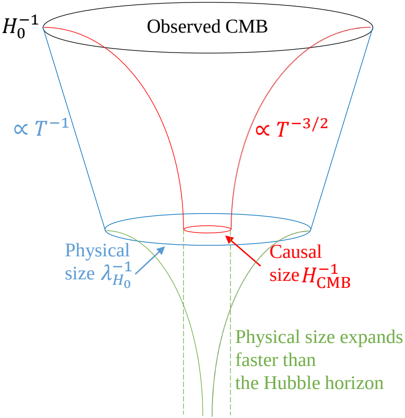
The homogeneity and isotropy cannot be explained

$$\frac{d}{dt}\left(\frac{\lambda}{H^{-1}}\right) \sim \frac{d}{dt}\left[\frac{a}{(\dot{a}/a)^{-1}}\right] = \ddot{a} < 0$$

Accelerated expansion: *Inflation*

$$\ddot{a} > 0\,,\ \text{e.g. cosmological constant } \Lambda : H = \text{constant}$$

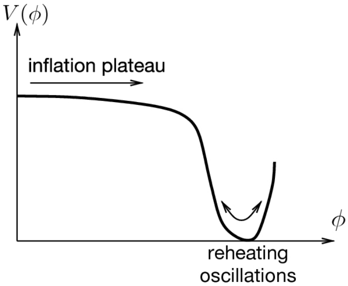




How to implement inflation?

$$S_m = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

$$\longrightarrow \frac{p}{\rho} = \frac{\frac{\dot\phi^2}{2} - V(\phi)}{\frac{\dot\phi^2}{2} + V(\phi)} \approx -1 \text{ if } V(\phi) \text{ dominates: c.c.}$$



Decomposition of cosmological perturbations

$$S=\int d^4x\sqrt{-g}\left[\frac{m_{\rm Pl}^2}{2}R-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)\right]$$

with $ds^2=-(1+2A)dt^2-2aB_idtdx^i+a^2(\delta_{ij}+2C_{ij})dx^idx^j$, $\phi=\phi_0+\delta\phi$

$$A=\alpha$$
$$B_i=\beta_{,i}+S_i$$
$$C_{ij}=\varphi\delta_{ij}+\gamma_{,ij}+F_{(i,j)}+\frac{1}{2}h_{ij}$$

What matter are scalar and tensor: constrained by observations

Quadratic action for cosmological perturbations (scalar)

After rescaling $u = z\mathscr{R} = \dfrac{a\phi_0'}{\mathcal{H}}\left(-\varphi + \dfrac{\mathcal{H}}{\phi_0'}\delta\phi\right)$, S_2 becomes

$$S_2 = \int d^4x \frac{1}{2}\left[u'^2 - (\nabla u)^2 + \frac{z''}{z}u^2\right]$$

Identical to the harmonic oscillator with time-dependent mass

$$u''(\tau, \boldsymbol{k}) + \left(k^2 - \frac{z''}{z}\right)u(\tau, \boldsymbol{k}) = 0$$

Standard quantization

We can follow the conventional wisdom to quantize perturbations

1. Promote to operators: $\widehat{u}(\tau,\boldsymbol{k}) = a_{\boldsymbol{k}} u_k(\tau) + a^\dagger_{-\boldsymbol{k}} u_k^*(\tau)$
2. Impose commutation relations: $\left[a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{k}-\boldsymbol{q})$
3. Define the vacuum state: $a_{\boldsymbol{k}}|0\rangle = 0$ for all $\boldsymbol{k}$

We find the Hankel function of the 1st kind as the solution

$$u_k(\tau) = \frac{\sqrt{-\pi\tau}}{2} e^{i(\nu+1/2)\pi/2} H_\nu^{(1)}(-k\tau) \underset{\nu=3/2}{=\!=} \frac{1}{\sqrt{2k}}\left(1-\frac{i}{k\tau}\right) e^{-ik\tau}$$

Power spectrum

$$\left\langle \mathscr{R}(\boldsymbol{k})\mathscr{R}(\boldsymbol{q})\right\rangle = (2\pi)^3\delta^{(3)}(\boldsymbol{k}+\boldsymbol{q})\frac{2\pi^2}{k^3}\mathscr{P}_{\mathscr{R}}(k)$$

Evaluated at the end of inf $(-k\tau \to 0)$ we find the well-known results

$$\mathscr{P}_{\mathscr{R}} = \left(\frac{H}{2\pi}\right)^2\left(\frac{H}{\dot{\phi}_0}\right)^2$$

$$n_{\mathscr{R}} - 1 = \frac{d\log\mathscr{P}_{\mathscr{R}}}{d\log k} = -2\epsilon - \eta$$

Tensor perturbations, a.k.a. GWs

We can follow almost identical steps for tensor perturbations

$$S_2 = \sum_\lambda \int d^4x \frac{1}{2}\left[v'^2 - (\nabla v)^2 + \frac{a''}{a}v^2\right] \qquad \left(v = \frac{am_{\rm Pl}}{\sqrt{2}}h\right)$$

$$v''(\tau, \boldsymbol{k}) + \left(k^2 - \frac{a''}{a}\right)v(\tau, \boldsymbol{k}) = 0$$

$$\mathcal{P}_h = \frac{8}{m_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2$$

$$n_h = -2\epsilon$$

"Consistency relation": $r \equiv \dfrac{\mathcal{P}_h}{\mathcal{P}_\mathcal{R}} = 16\epsilon = -8n_h$

Hamiltonian with coupled variables

$$S_2 = \int d^4x \frac{1}{2} \left[u'^2 - (\nabla u)^2 + \underbrace{\frac{a''}{a} u^2}_{} \right]$$

$$= \frac{a'}{2a^2} u^2 \Big|_{\text{boundary}} - \int \left[\frac{a'}{a} u u' - \frac{1}{2} \left(\frac{a'}{a} \right)^2 u^2 \right]$$

$$= \int d^4x \frac{1}{2} \left[u'^2 - 2\frac{a'}{a} u' u - (\nabla u)^2 + \left(\frac{a'}{a} \right)^2 u^2 \right]$$

Thus the Hamiltonian H couples conjugate momentum π and u

$$H = \int d^3x \mathscr{H} = \int d^3x \left(\pi u' - \mathscr{L} \right)$$

$$= \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[\underbrace{\pi_{\mathbf{k}} \pi_{-\mathbf{k}} + k^2 u_{\mathbf{k}} u_{-\mathbf{k}}}_{\text{free Hamiltonian}} + \underbrace{\frac{a'}{a} \left(\pi_{\mathbf{k}} u_{-\mathbf{k}} + u_{\mathbf{k}} \pi_{-\mathbf{k}} \right)}_{\text{interacting Hamiltonian}} \right]$$

Harmonic oscillator decomposition

We promote $\pi_{\boldsymbol{k}}$ and $u_{\boldsymbol{k}}$ to the (time-dependent) operators $\widehat{\pi}_{\boldsymbol{k}}$ and $\widehat{u}_{\boldsymbol{k}}$

$$\widehat{\pi}_{\boldsymbol{k}} = a_{\boldsymbol{k}}(\tau)\pi_k + a^\dagger_{-\boldsymbol{k}}(\tau)\pi^*_k$$
$$\widehat{u}_{\boldsymbol{k}} = a_{\boldsymbol{k}}(\tau)u_k + a^\dagger_{-\boldsymbol{k}}(\tau)u^*_k$$

N.B. Time dependence to $a_{\boldsymbol{k}} = a_{\boldsymbol{k}}(\tau)$ and $a^\dagger_{\boldsymbol{k}} = a^\dagger_{\boldsymbol{k}}(\tau)$, mode fct fixed

Initial condition: the expectation value of $\widehat{H}_0$ is minimized at $\tau = \tau_0$ (taken w.r.t. initial vacuum state $a_{\boldsymbol{k}}|0\rangle = 0$ for all $\boldsymbol{k}$)

$$\widehat{\pi}_{\boldsymbol{k}} = -i\sqrt{\frac{k}{2}}\left[a_{\boldsymbol{k}}(\tau) - a^\dagger_{-\boldsymbol{k}}(\tau)\right]$$
$$\widehat{u}_{\boldsymbol{k}} = \frac{a_{\boldsymbol{k}}(\tau) + a^\dagger_{-\boldsymbol{k}}(\tau)}{\sqrt{2k}}$$

Evolution equation for operators

From Hamiltonian equations (suppressing time dep for simplicity)

$$a'_{\boldsymbol{k}}(\tau) = -ika_{\boldsymbol{k}}(\tau) + \frac{a'}{a}a^{\dagger}_{-\boldsymbol{k}}(\tau)$$

$$a^{\dagger}_{-\boldsymbol{k}}{}'(\tau) = ika^{\dagger}_{-\boldsymbol{k}}(\tau) + \frac{a'}{a}a_{\boldsymbol{k}}(\tau)$$

General sol given by the linear combinations of the initial ones:

$$a_{\boldsymbol{k}}(\tau) = \alpha_k(\tau)a_{\boldsymbol{k}}(\tau_0) + \beta_k(\tau)a^{\dagger}_{-\boldsymbol{k}}(\tau_0)$$

$$a^{\dagger}_{-\boldsymbol{k}}(\tau) = \alpha^*_k(\tau)a^{\dagger}_{-\boldsymbol{k}}(\tau_0) + \beta^*_k(\tau)a_{\boldsymbol{k}}(\tau_0)$$

"Bogoliubov transformation"

Parametrization of the solutions

From the standard commutation relations between $a_{\boldsymbol{k}}(\tau)$ and $a_{\boldsymbol{k}}^\dagger(\tau)$

$$\left[a_{\boldsymbol{k}}, a_{\boldsymbol{q}}^\dagger\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{q})\underbrace{\left(|\alpha_k|^2 - |\beta_k|^2\right)}_{\text{should be 1 always}}$$

Thus $\alpha_k(\tau)$ and $\beta_k(\tau)$ can be parametrized i.t.o. hyperbolic fcts

$$\alpha_k = e^{-i\theta_k}\cosh r_k$$
$$\beta_k = e^{i(\theta_k+2\varphi_k)}\sinh r_k$$

Without loss of generality, we take t-dep $r_k(\tau)$, $\theta_k(\tau)$ and $\varphi_k(\tau)$ real

Solutions for the transformation parameters

Evolution equations for $r_k(\tau)$, $\theta_k(\tau)$ and $\varphi_k(\tau)$ (*difficult to obtain!*)

$$r_k' = \frac{a'}{a}\cos(2\varphi_k)$$

$$\theta' = k + \frac{a'}{a}\sin(2\varphi_k)\tanh r_k$$

$$\varphi' = -k - \frac{a'}{a}\sin(2\varphi_k)\coth(2r_k)$$

dS solutions ($a'/a = -1/\tau$) are found (*again difficult to obtain!*)

$$r_k = \sinh^{-1}\left(\frac{1}{2k\tau}\right)$$

$$\varphi_k = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{1}{2k\tau}\right)$$

$$\theta_k = k\tau + \tan^{-1}\left(\frac{1}{2k\tau}\right)$$

Hamiltonian discretized

Putting the annihilation operator $a_{\boldsymbol{k}}$ on the right, $\widehat{\mathscr{H}}$ is written as

$$\widehat{\mathscr{H}}(\tau) = \frac{1}{2}\left\{k\left[(2\pi)^3\delta^{(3)}(\boldsymbol{0}) + a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} + a^\dagger_{-\boldsymbol{k}}a_{-\boldsymbol{k}}\right] + i\frac{a'}{a}\left[-a_{\boldsymbol{k}}a_{-\boldsymbol{k}} + a^\dagger_{-\boldsymbol{k}}a^\dagger_{\boldsymbol{k}}\right]\right\}$$

For a single k-mode, we need $\widehat{\mathscr{H}}$, not necessarily $\widehat{H} = \int d^3k/(2\pi)^3\widehat{\mathscr{H}}$
"Discretization" of momenta: $\int d^3k/(2\pi)^3 \to L^{-3}\sum_k$

$$a_{\boldsymbol{k}} \equiv L^{3/2}\hat{a}_{\boldsymbol{k}} \quad \text{so that} \quad \left[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}}\right] = 1$$

$$\widehat{\mathscr{H}} = L^3 \times \frac{1}{2}\left\{k\left[1 + \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right] + i\frac{a'}{a}\left[-\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]\right\} \equiv L^3\widehat{\mathscr{H}}_k$$

$$\widehat{H} = L^{-3}\sum_k \widehat{\mathscr{H}} = \sum_k \widehat{\mathscr{H}}_k$$

Factorized time-evolution operator

The time-evolution operator is factorized as

$$\hat{U}_{\boldsymbol{k}} = \underbrace{\exp\left[-i\theta_k\left(1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)\right]}_{\equiv\hat{R}_{\boldsymbol{k}}}\underbrace{\exp\left[\frac{r_k}{2}\left(\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}e^{-2i\varphi_k}-\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}e^{2i\varphi_k}\right)\right]}_{\equiv\hat{S}_{\boldsymbol{k}}}$$

$\hat{R}_{\boldsymbol{k}}$: “Rotation” operator

$\hat{S}_{\boldsymbol{k}}$: “Squeezing” operator

What the rotation operator does

We can operate $\widehat{U}_{\boldsymbol{k}}$ solely onto initial vacuum $|0\rangle$ (Schrödinger pic)

$$\widehat{R}_{\boldsymbol{k}}|0\rangle = \left[1 - i\theta_k\left(1 + \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right) + \frac{1}{2}(-i\theta_k)^2\left(1 + \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)^2 + \cdots\right]|0\rangle$$
$$= \left[1 - i\theta_k + \frac{1}{2}(-i\theta_k)^2 + \cdots\right]|0\rangle$$
$$= e^{-i\theta_k}|0\rangle$$

- $\widehat{R}_{\boldsymbol{k}}$ produces an irrelevant phase θ_k (“rotation”)
- $\widehat{R}_{\boldsymbol{k}}$ comes from free Hamiltonian, no expansion is considered
- Same as in Minkowski space-time: Vacuum remains vacuum

What the squeezing operator does

We can rewrite $\widehat{S}_{\boldsymbol{k}}$ as (*not as difficult as it seems!*)

$$\widehat{S}_{\boldsymbol{k}} = \exp\left[-e^{2i\varphi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right]\left[\frac{1}{\cosh(r_k/2)}\right]^{1+\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}} \times \exp\left[e^{2i\varphi_k}\tanh\left(\frac{r_k}{2}\right)\hat{a}_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\right]$$

From the Project [$\alpha_0 = 0$, $\alpha_- = -\alpha_+ = r_k/2$ and $\gamma_\pm = \mp\tanh(r_k/2)$]

$$\widehat{S}_{\boldsymbol{k}}|0\rangle = \frac{1}{\cosh(r_k/2)}\sum_{n=0}^{\infty}\left[-e^{2i\varphi_k}\tanh\left(\frac{r_k}{2}\right)\right]^n \underbrace{\frac{1}{n!}\left(\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right)^n|0\rangle}_{\equiv|n,\boldsymbol{k};n.-\boldsymbol{k}\rangle}$$

- Creation of 2 quanta w/ $\boldsymbol{k}$ and $-\boldsymbol{k}$ (momentum conservation)
- 2-mode state w/ same occupation number n (n-ptl state)
- Cosmological perturbations are amplified

Conclusions

- Cosmological perturbations can explain the observed structure
- Quantum fluctuations leads to the cosmological perturbations
- Gravity is responsible for the generation of perturbations
- Vacuum state evolves into a non-trivial quantum state
- Many interesting studies ahead