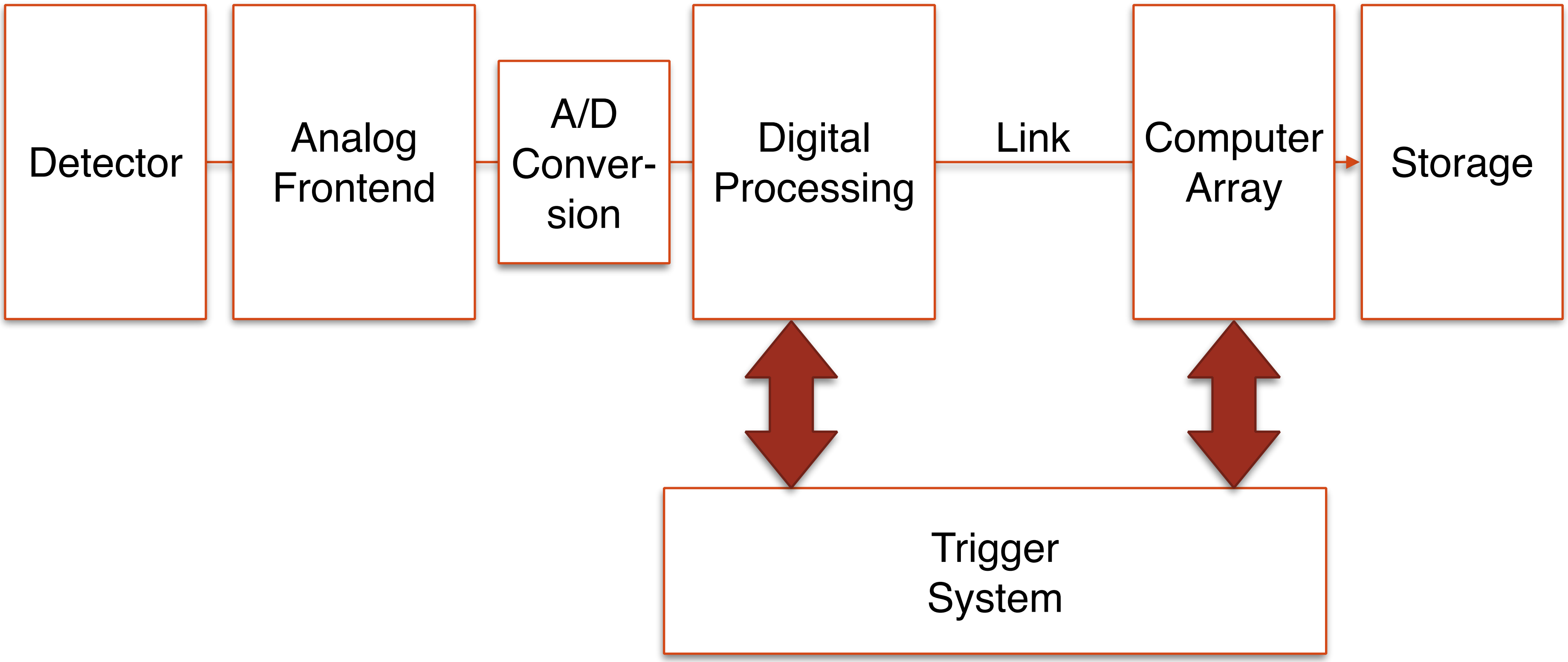
Mathmatics for Frontend Circuit Design

21/Jan/2022

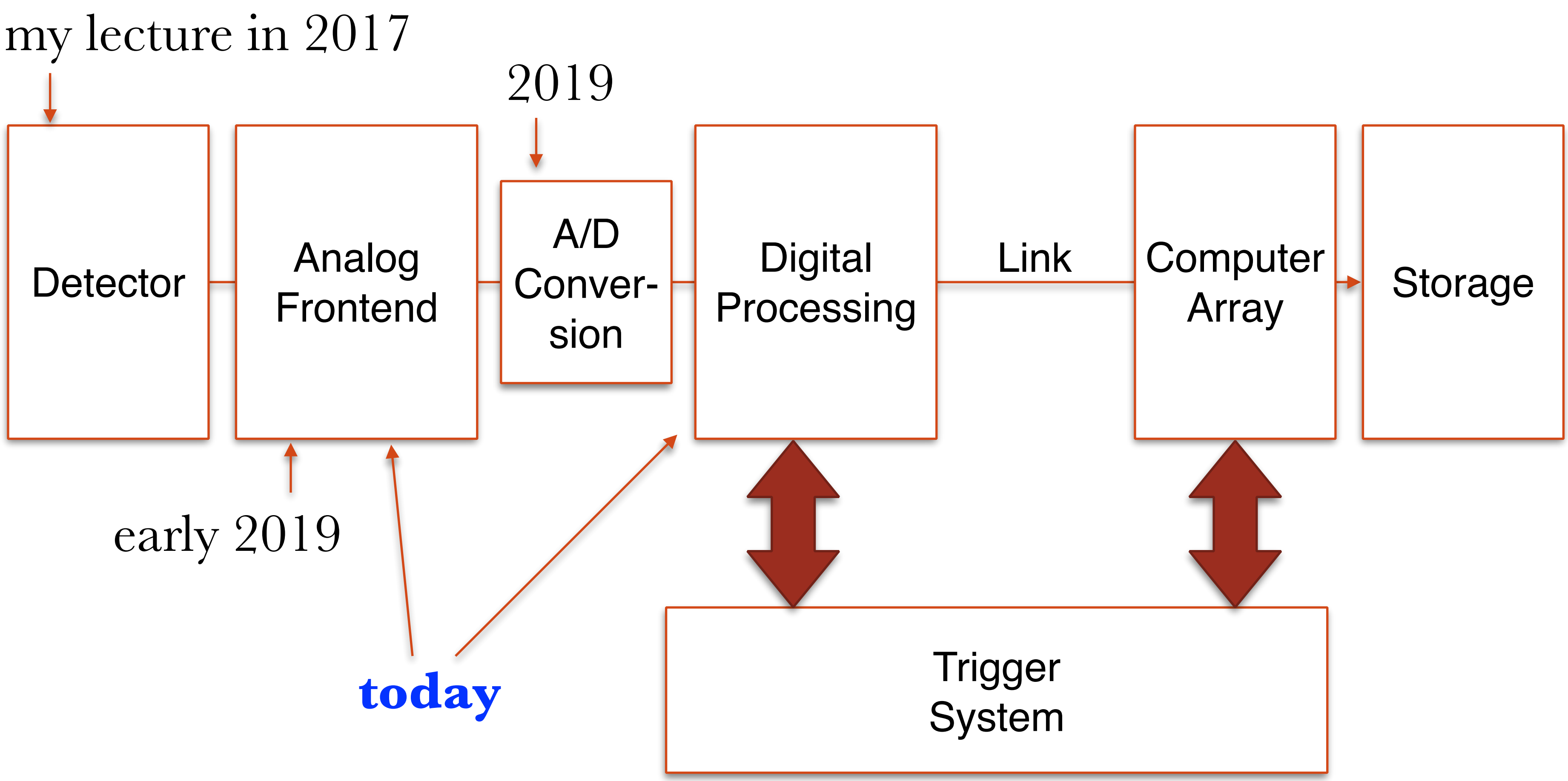
The 18th Yonsei-Saga Workshop

Takahiro Fusayasu @Saga Univ.

From Detector to DAQ



From Detector to DAQ



Outline

- Frontend Analog Circuits for Gas Detectors
  - Preparations
    - Laplace Transform
    - Operational Amplifier
  - Example of Frontend Circuits and its Analysis
- z-transform for digital filtering
- Summary

Frontend Analog Circuits for Gas Detectors

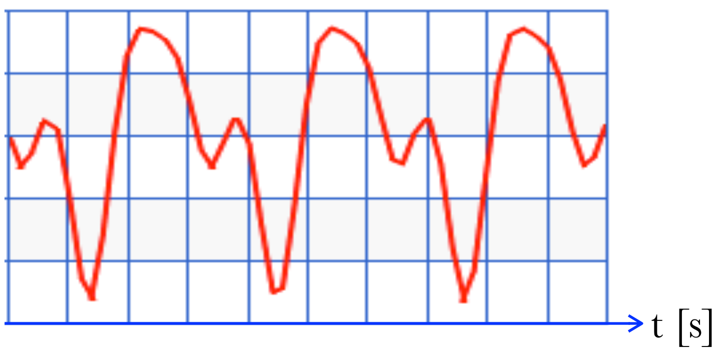
Background Knowledge

- Laplace transform
  Transient properties of analog circuits are usually expressed by differential equations. To solve the differential equations easily, Laplace transform is a powerful tool.

- OP amps (Operational Amplifiers)
  Amplifier circuits, adder circuits, differentiator, Integrator…
  various operational circuits can be composed of OP amps.

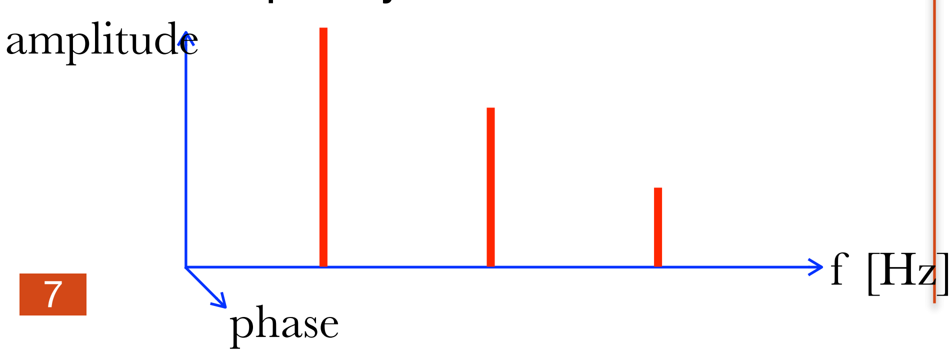
What is Laplace Transform?
In comparison with Fourier Transform

periodic wave in time domain

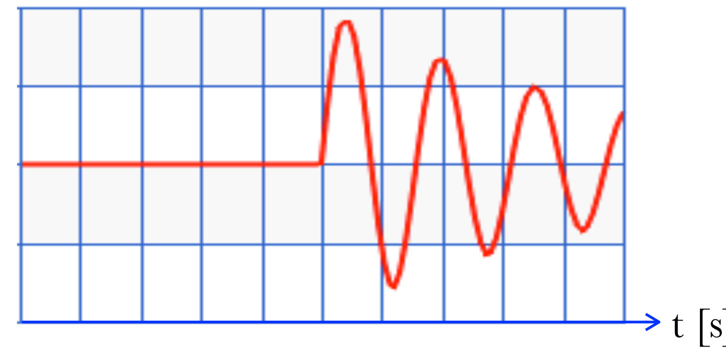


Fourier transform ↓ ↑ inverse Fourier transform

frequency domain

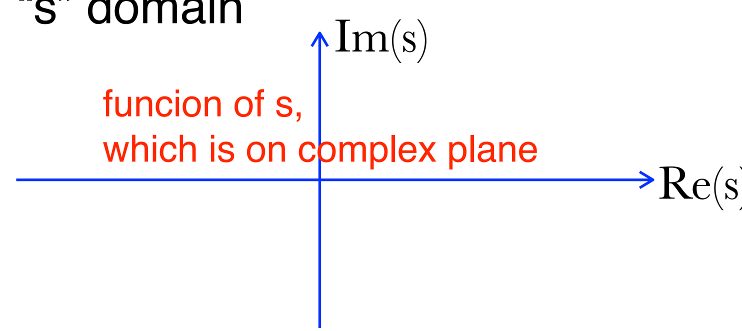


non-periodic function in time domain



Laplace transform ↓ ↑ inverse Laplace transform

“s” domain



Circuit Analysis with Laplace Transform

- Definition of Laplace transform

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

- Differential is transformed to s and integration is transformed to 1/s. $\rightarrow$ Electronic circuit problems are solved by arithmetic operations ($+ - \times \div$)

- How to solve circuit problems:
  1. A circuit (t-domain) is Laplace transformed to an s-domain circuit.
  2. The obtained circuit is solved in the s-domain using the Ohm's law and Kirchhoff's lows.
  3. The obtained solution is inverse-Laplace transformed to t-domain.

Table of Laplace Transforms

- We don’t need to calculate the integration shown before by ourselves. Instead, distributed tables are used to obtain Laplace transforms.
- Pickups from the Laplace transform table.
  Left → Right : Laplace transform,
  Left ← Right : Inverse Laplace transform.

t=0

	f(t)	F(s)
unit impulse function	$\delta(t)$	1
unit step function	$u(t)$	$1/s$
	$e^{-at}u(t)$	$1/(s+a)$
	$df(t)/dt$	$sF(s) - f(0)$
	$\int f(t)dt$	$(1/s)F(s)$
Time shift	$f(t-T)$	$e^{-sT}F(s)$

Laplace Transform of a Capacitor

According to the Laplace transform table, differential is transformed as

$$\frac{df(t)}{dt} \Leftrightarrow sF(s) - f(0)$$

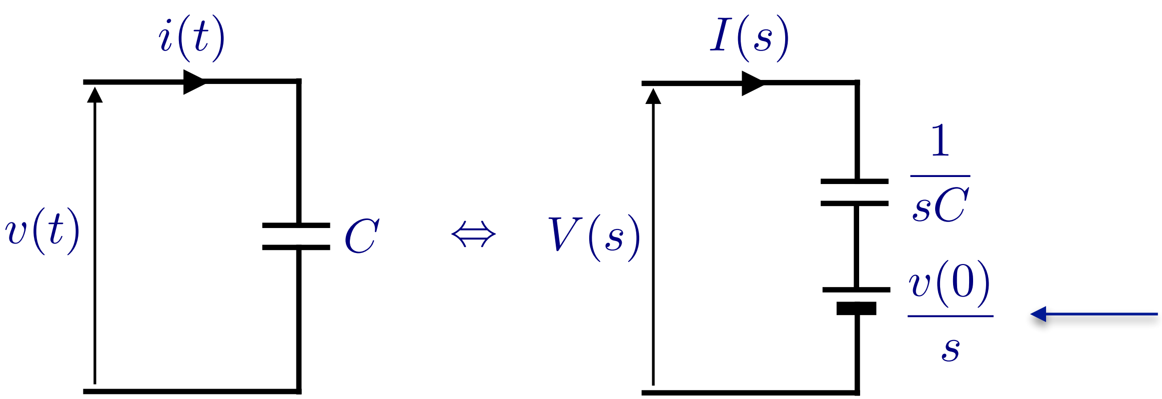
with which, capacitor's v-i relation in the time domain,

$$i(t) = C\frac{dv(t)}{dt}$$

is transformed to an s-domain equation:

$$I(s) = C\left(sV(s) - v(0)\right) = sCV(s) - Cv(0) \qquad \therefore V(s) = \frac{1}{sC}I(s) + \frac{v(0)}{s}$$

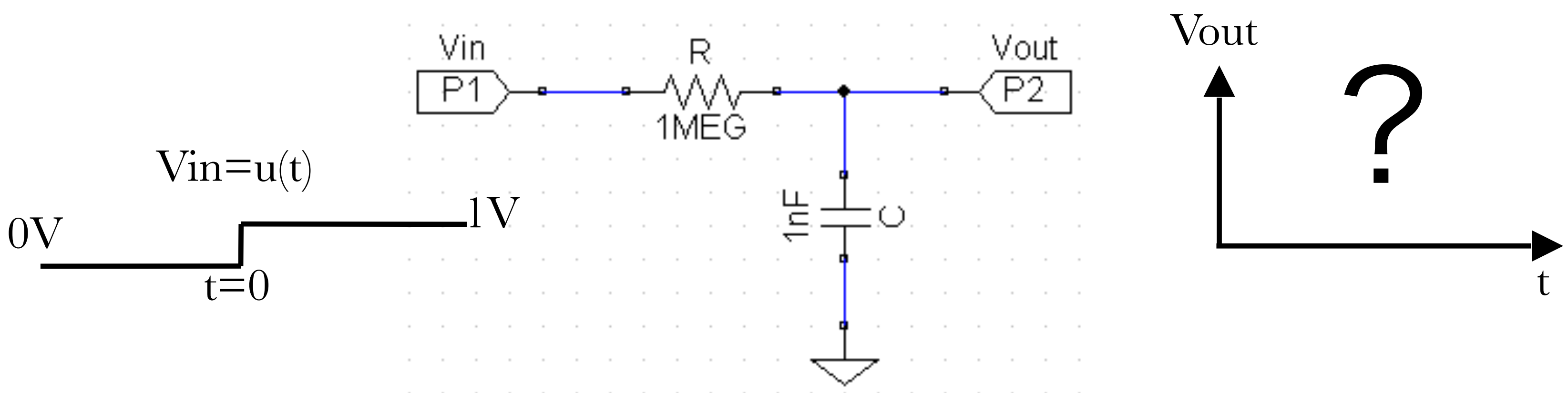
where V(s) and I(s) are Laplace transforms of v(t) and i(t). As a conclusion, a capacitor C is expressed as a passive device with impedance 1/sC in the s-domain circuit if the initial voltage v(t=0) is zero.



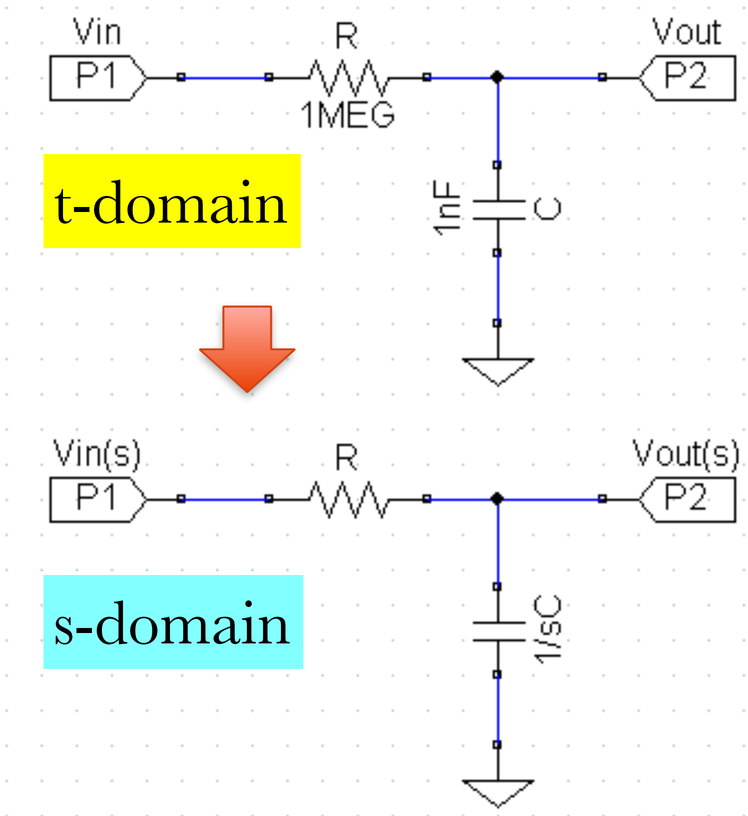
In many cases, there is no charge in the initial state therefore v(0)=0. Then the battery can be removed.

Example:
Analysis of RC Low-Pass Filter by Laplace Transform

- A step function with a step of 1V is applied to the voltage input of the RC circuit below. Give the output voltage shape.



Example:
Analysis of RC Low-Pass Filter by Laplace Transform



Analogous to the resistor attenuator,

$$V_{out} = \frac{R_2}{R_1 + R_2} V_{in}$$

s-domain in-out property of the RC LPF is

$$V_{out}(s) = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} V_{in}(s) = \frac{1}{1 + sRC}\frac{1}{s}$$

$$= \frac{1}{s} - \frac{1}{s + \frac{1}{RC}}$$

From the Laplace transform table,

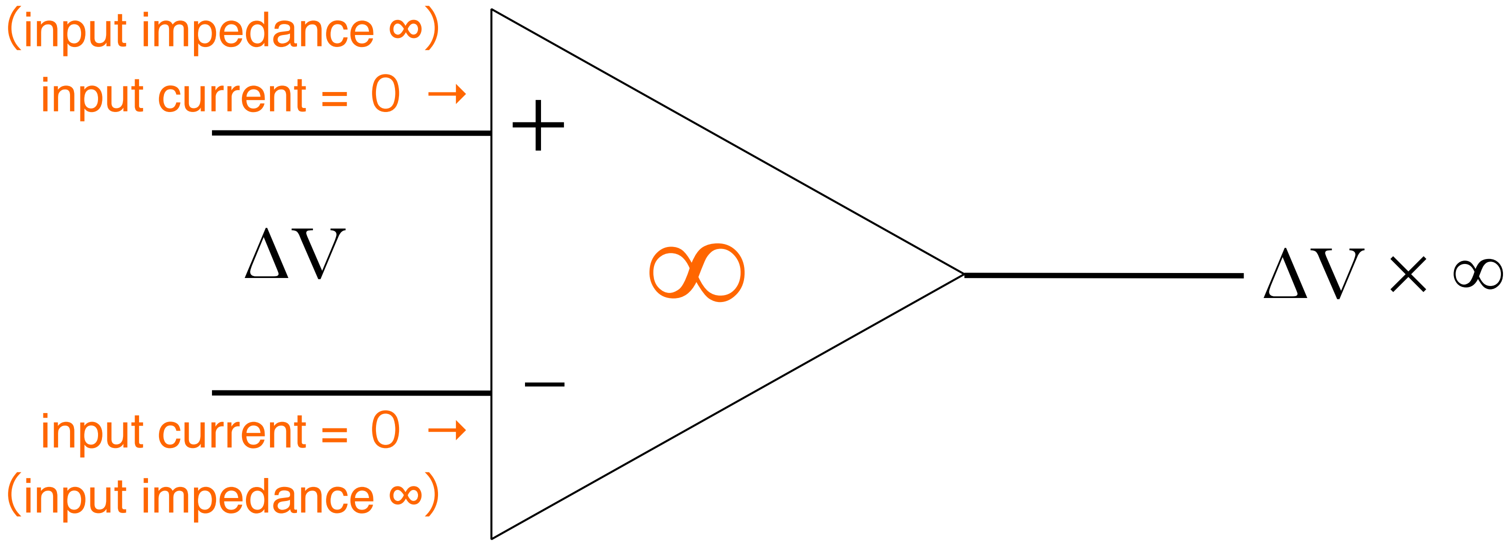
$$F(s) \to f(t) : \quad \frac{1}{s} \to u(t), \quad \frac{1}{s+a} \to e^{-at}u(t)$$

Therefore we obtain:

$$V_{out}(t) = u(t) - e^{-\frac{t}{RC}}u(t) = 1 - e^{-\frac{t}{RC}} \, (t > 0)$$

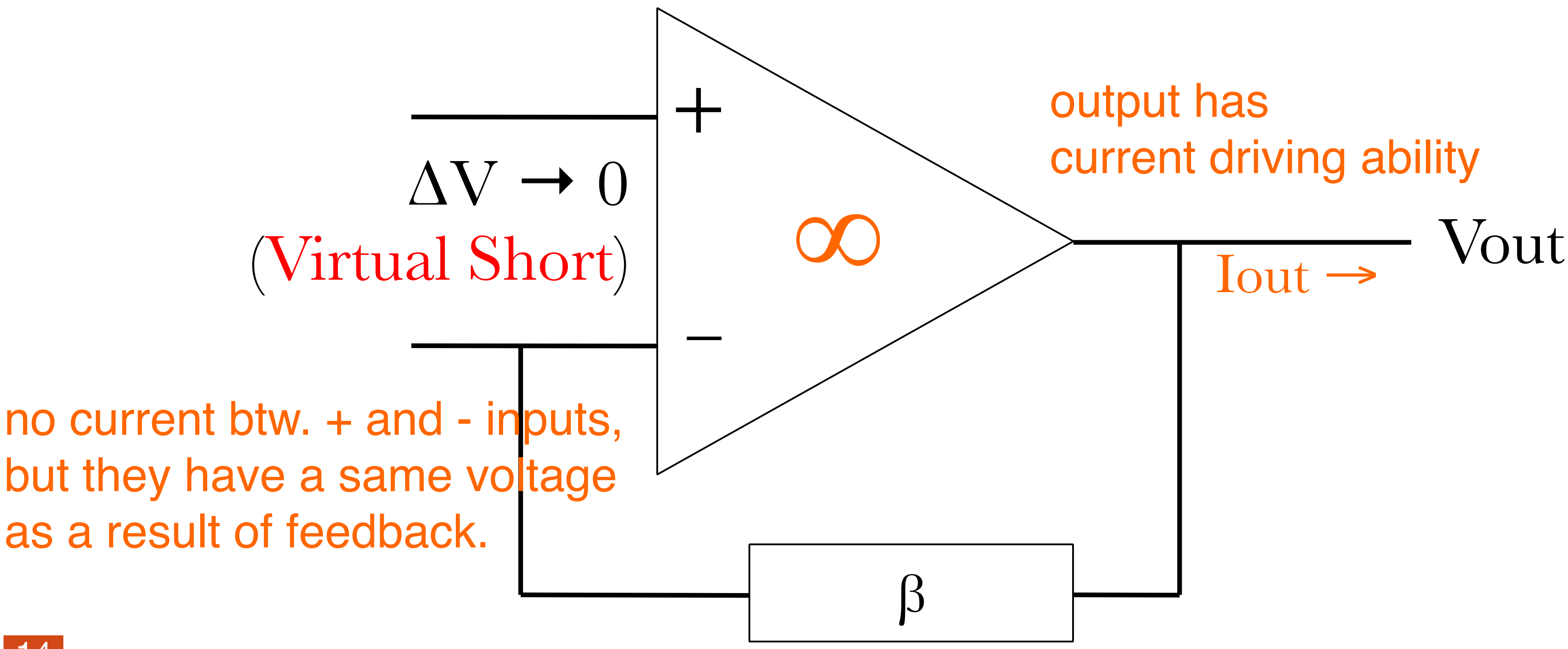
Ideal OP amp
(OP amp: Operational Amplifier)

- Infinite amplification of differential input.

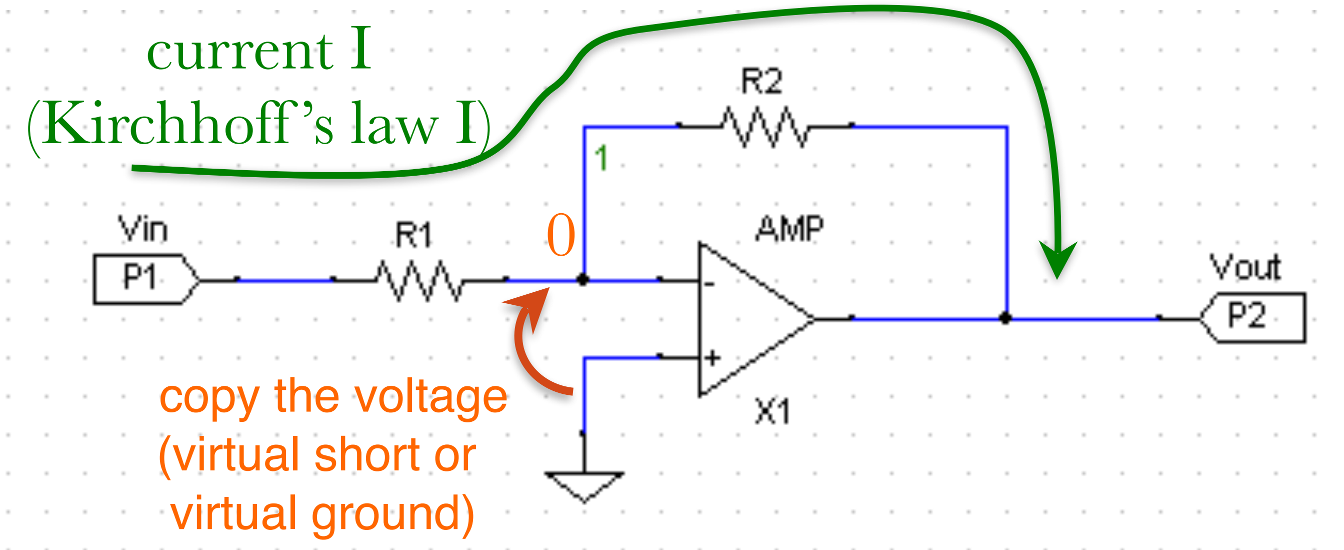


Negative Feedback of OP amps

- Feedback of output to the negative input makes the output voltage finite.



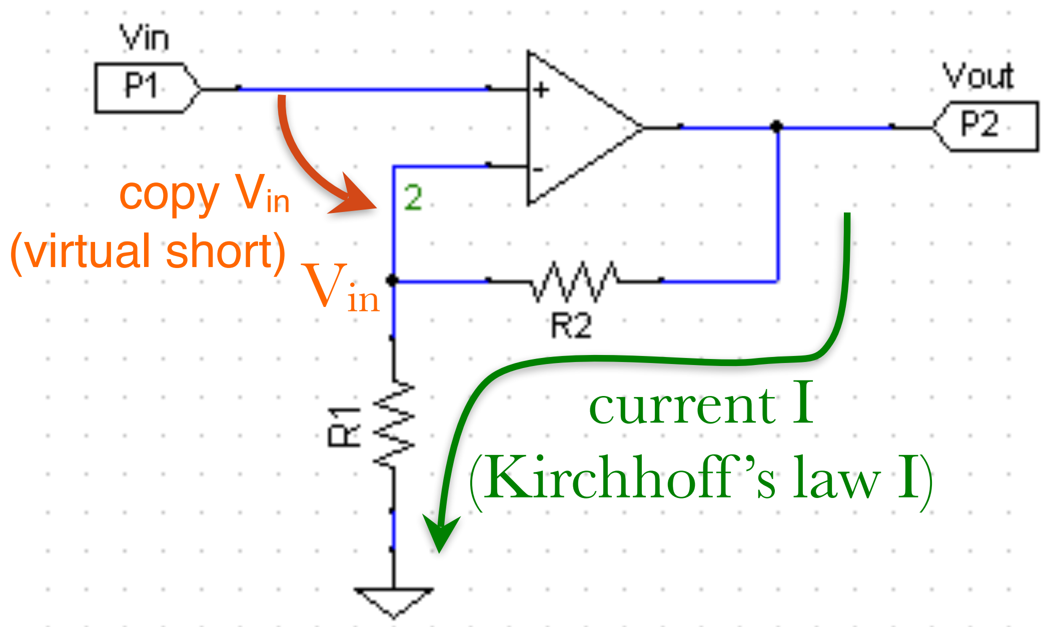
Basic OP amp circuit 1: Inverting Amplifier circuit



$$I = \frac{V_{in} - 0}{R} = \frac{V_{in}}{R}$$ (Ohm's law for R_1)

$$V_{out} = 0 - IR_2 = -\frac{R_2}{R_1}V_{in}$$ (Ohm's law for R_2)

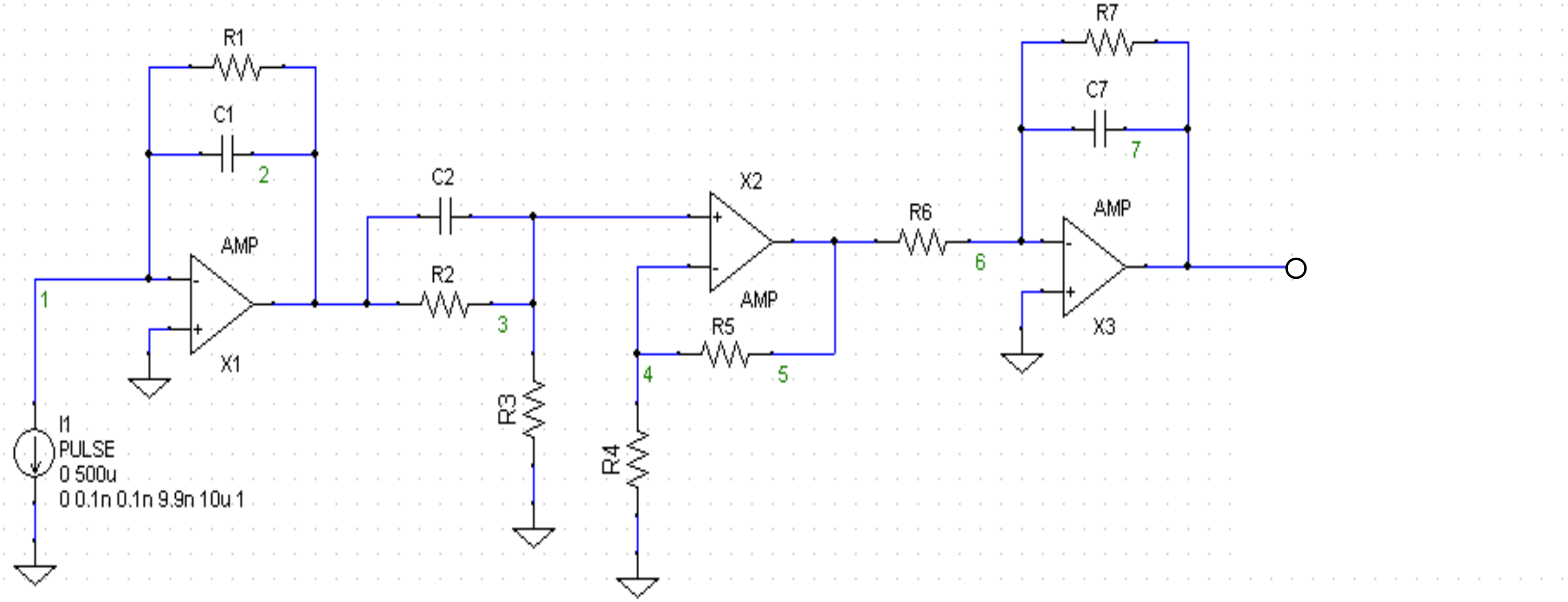
Basic OP amp circuit 2: Non-Inverting Amplifier circuit



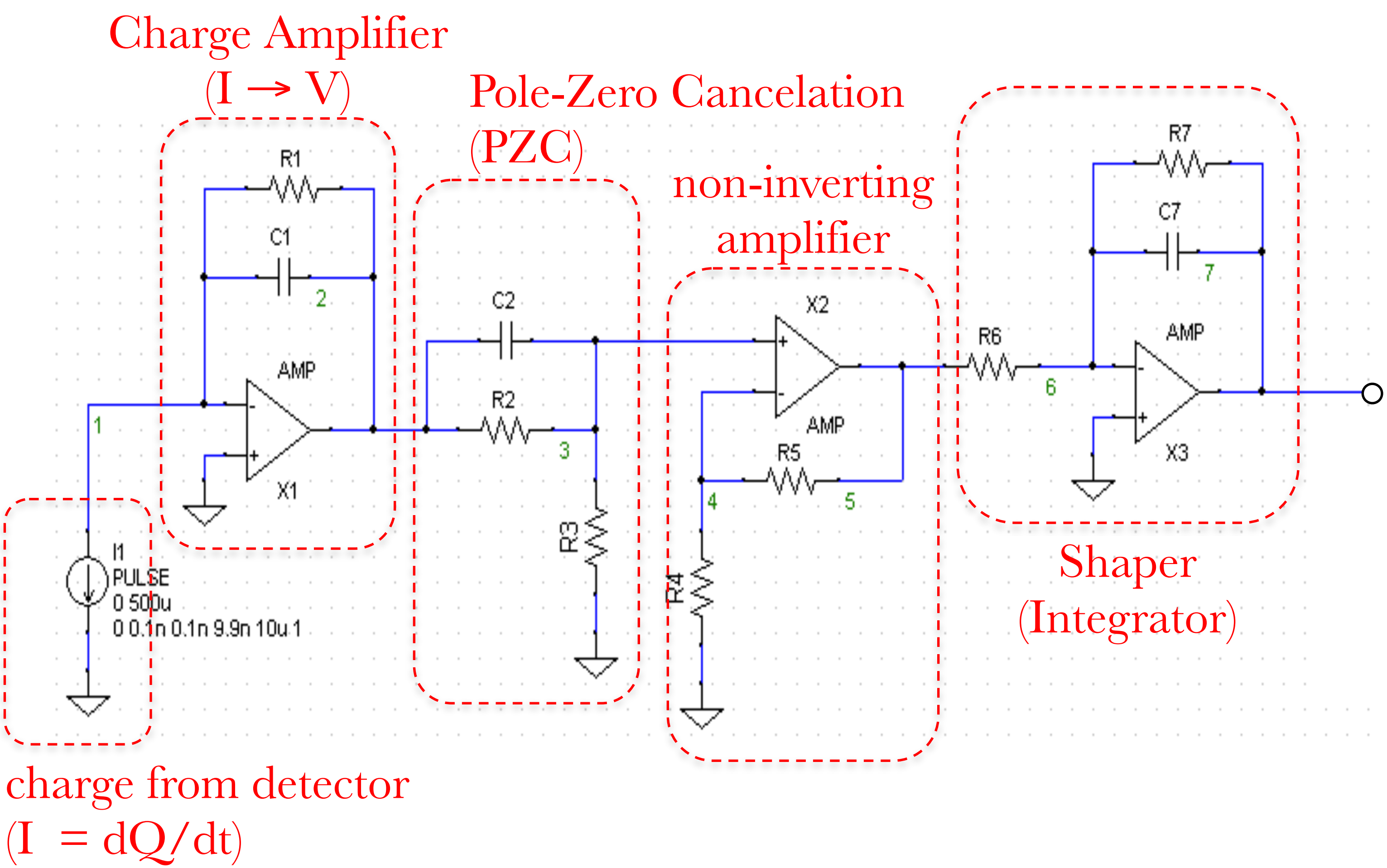
V_{in} is copied to the negative input by the virtual short theory. Then, the resistor chain works as an attenuator when you look it from the output of the OP amp.

$$V_{in} = \frac{R_1}{R_1 + R_2} V_{out} \qquad \therefore V_{out} = \left(1 + \frac{R_2}{R_1}\right) V_{in}$$

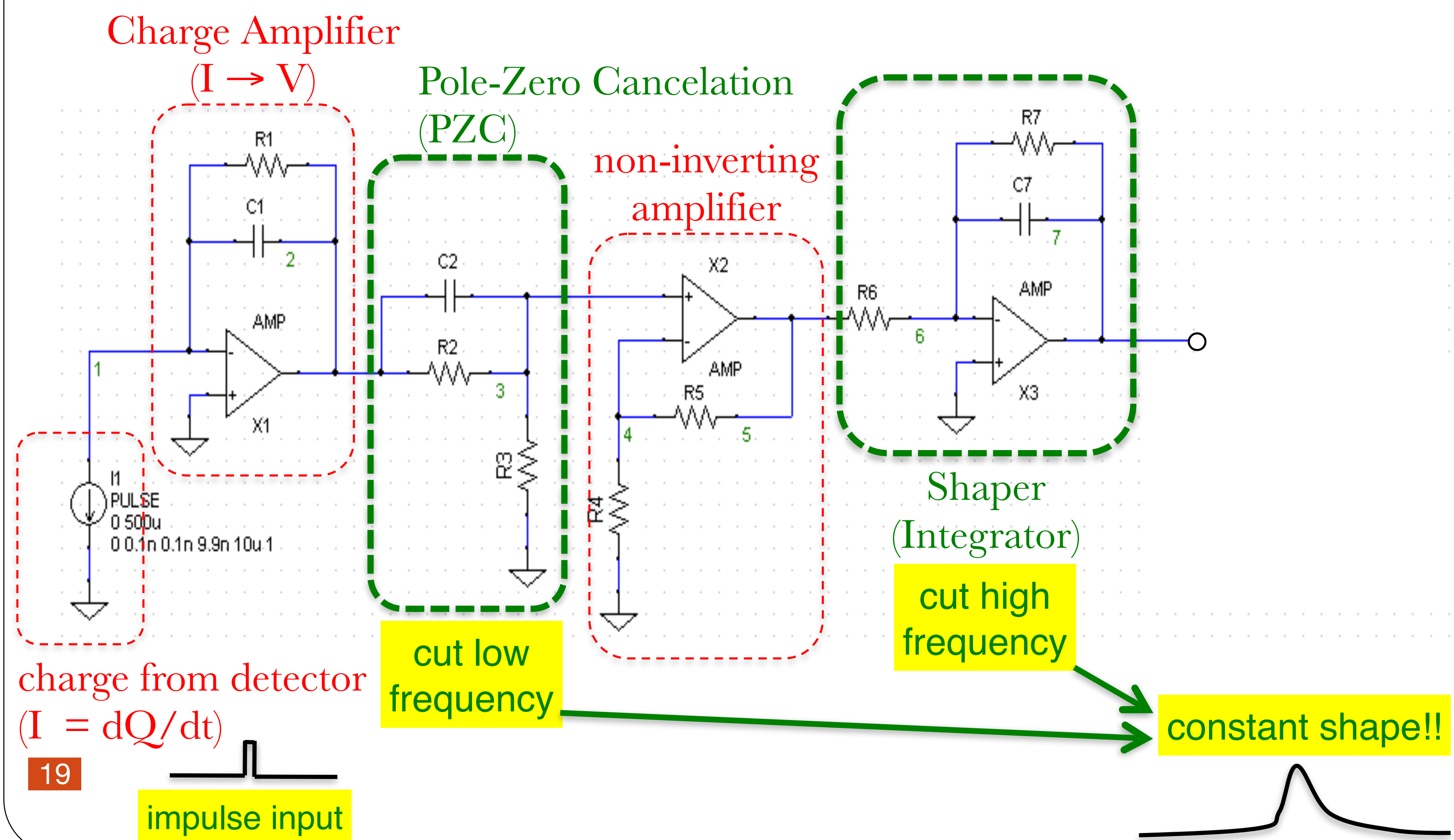
Example of Analog Frontend circuit for Gas Detectors



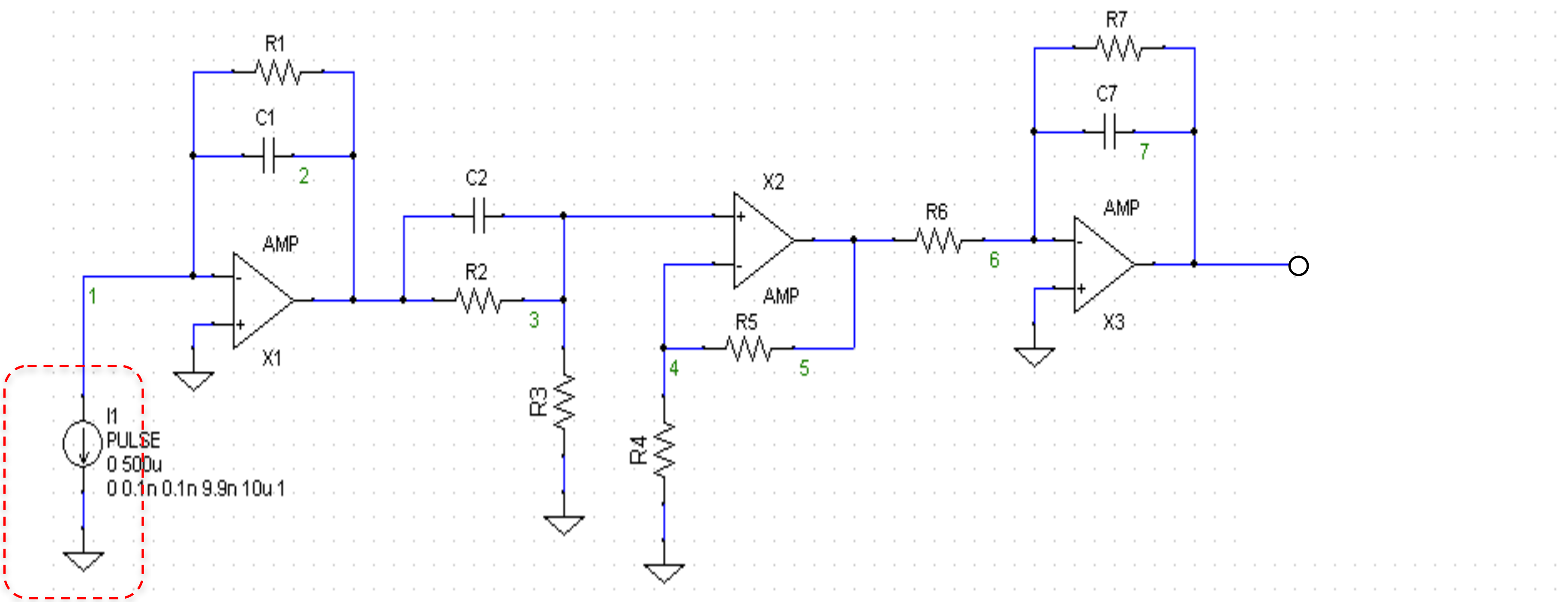
Example of Analog Frontend circuit for Gas Detectors



Example of Analog Frontend circuit for Gas Detectors



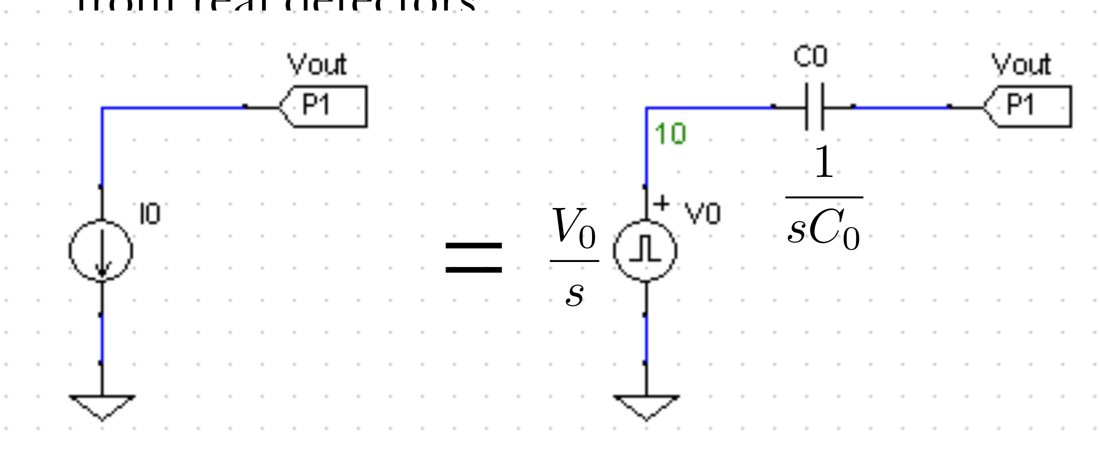
Signal Source



charge from detector
(I = dQ/dt)

Signal Source

- Signals from detectors are "charge" or "current pulse."
- Instead of a real detector, you can generate test pulses by applying voltage steps to a capacitor.
- Because cyclic pulses are composed of repetitive rising and falling edges, positive and negative charges are alternately injected, which is different from real detectors.



$$\delta(t) \Leftrightarrow 1$$

$$u(t) \Leftrightarrow \frac{1}{s}$$

$$C \Leftrightarrow \frac{1}{sC}$$

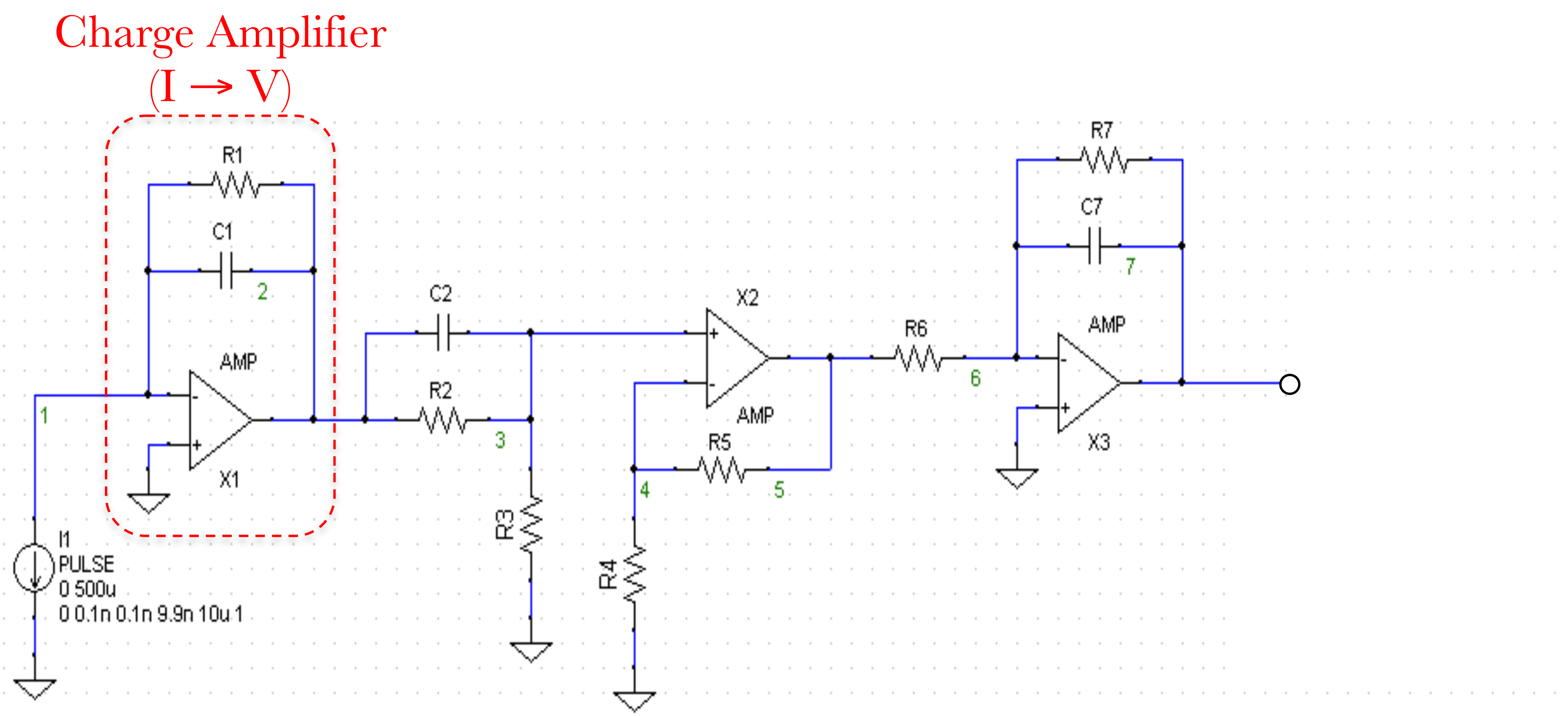
$$I_0(s) = Q$$

$$i_0(t) = Q\delta(t)$$

$$I_0(s) = \frac{V_0}{s} \div \frac{1}{sC_0} = C_0 V_0$$

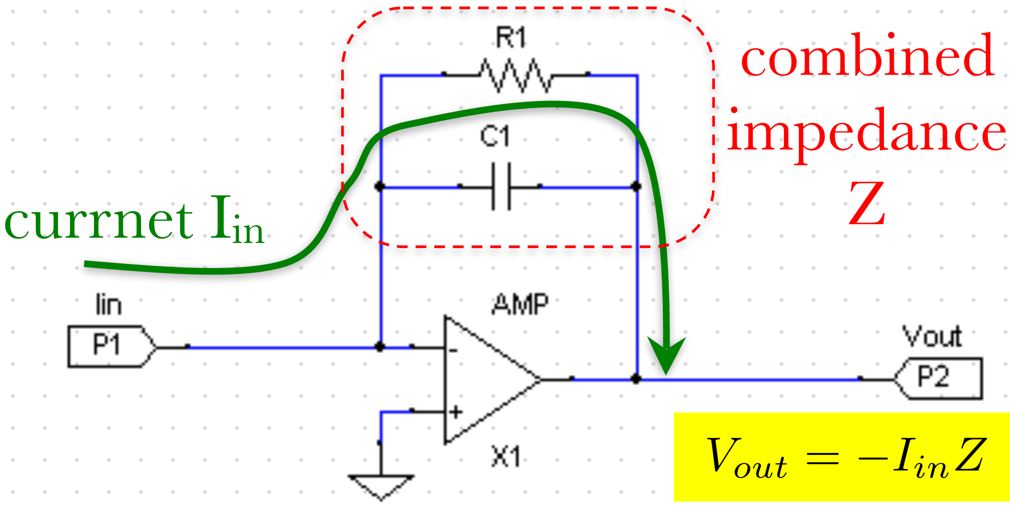
$$i_0(t) = C_0 V_0 \delta(t)$$

Charge Amplifier

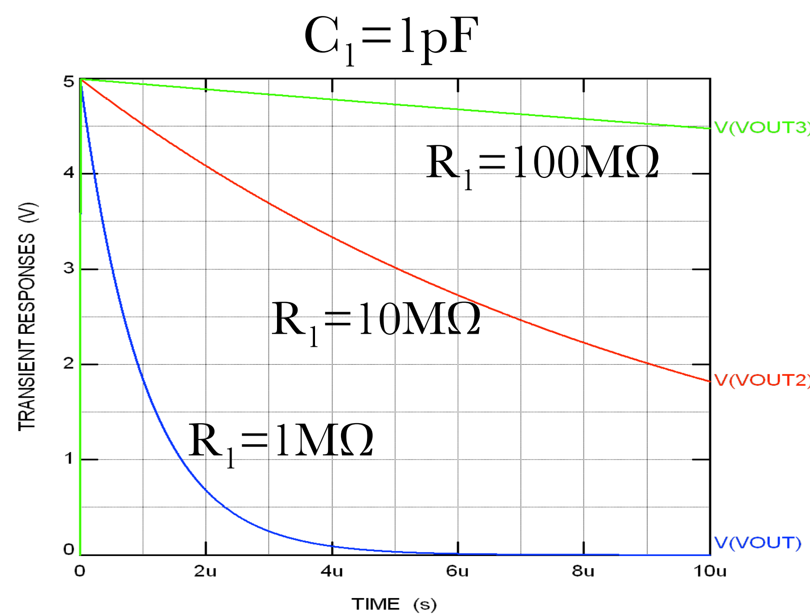


Charge Amplifier

- Convert the input charge to voltage. C_1 determines the gain and the time constant C1R1 defines the recovery time.



$V_{out} = -I_{in} Z$



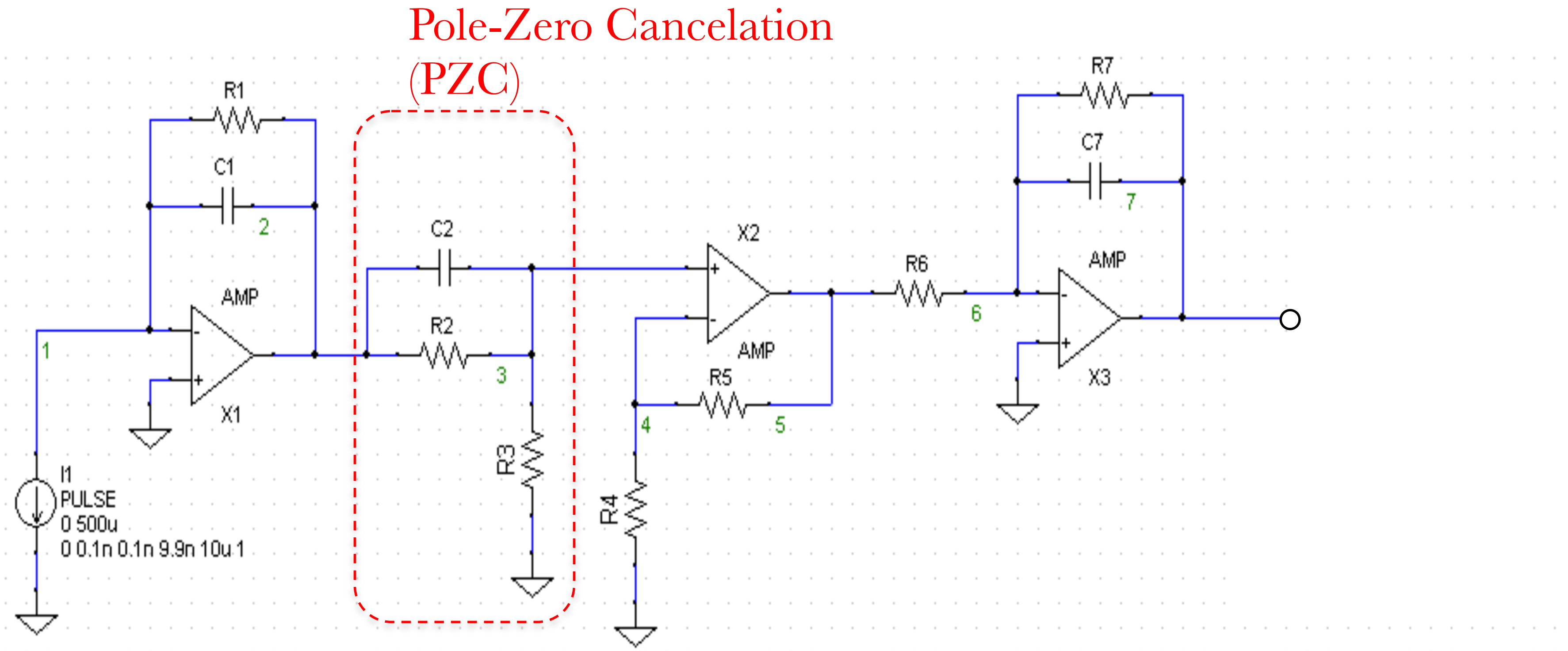
transfer function

$$T_1(s) \equiv \frac{V_{out}(s)}{I_{in}(s)} = -R_1 \parallel \frac{1}{sC_1} = -\frac{R_1}{1+sC_1R_1}$$

$$= -\frac{1}{C_1}\frac{1}{s+\frac{1}{C_1R_1}} \Leftrightarrow -\frac{1}{C_1}\exp\left(-\frac{1}{C_1R_1}t\right)$$

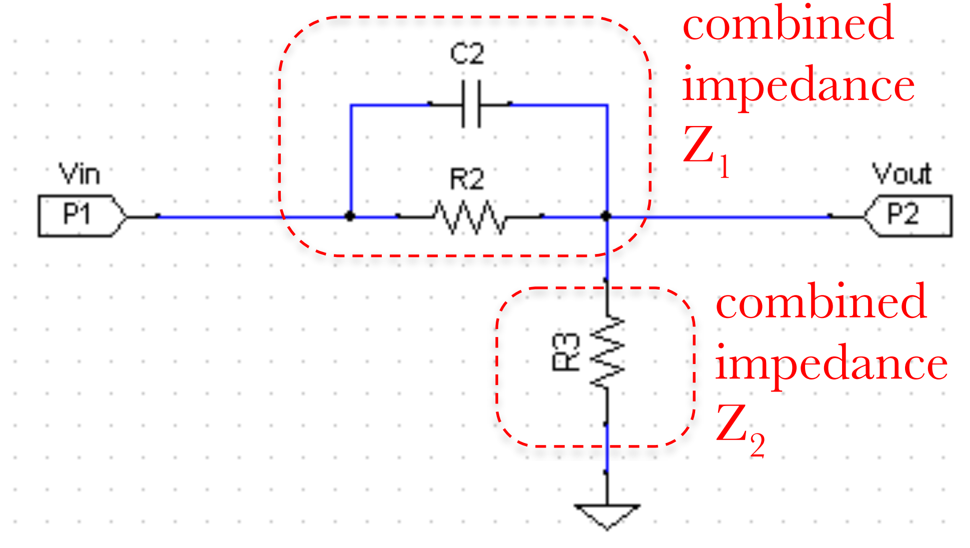
$$e^{\alpha t} \Leftrightarrow \frac{1}{s-\alpha}$$

Pole-Zero Cancellation (PZC)



Pole-Zero Cancellation (PZC)

- By setting the time constant same as the previous stage T_1, i.e. $C_1R_1 = C_2R_2$, the pole of T_1 and zero of T_2 are canceled. This is called pole-zero cancellation (PZC).

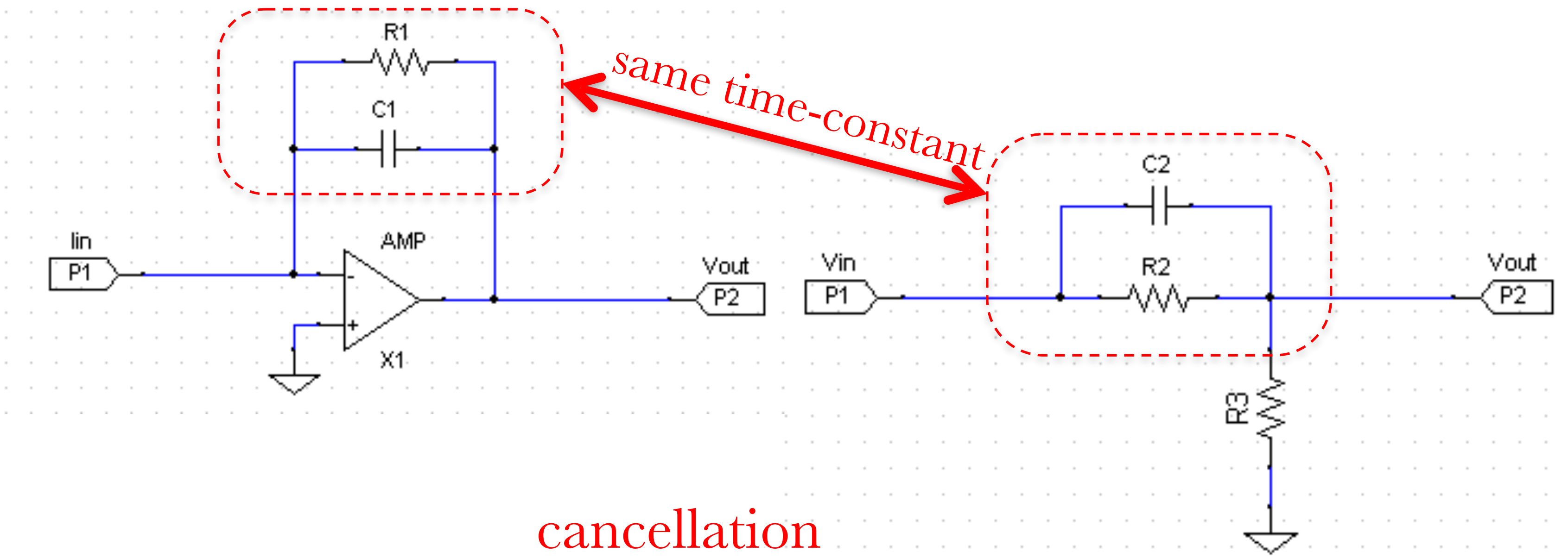


attenuator consists of Z_1 and Z_2

$$T_2(s) \equiv \frac{V_{out}(s)}{V_{in}(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{R_3}{\left(R_2 || \frac{1}{sC_2}\right) + R_3} = \frac{R_3}{R_2 + R_3} \frac{1 + sC_2R_2}{1 + sC_2(R_2||R_3)}$$

Pole-Zero Cancellation (PZC)

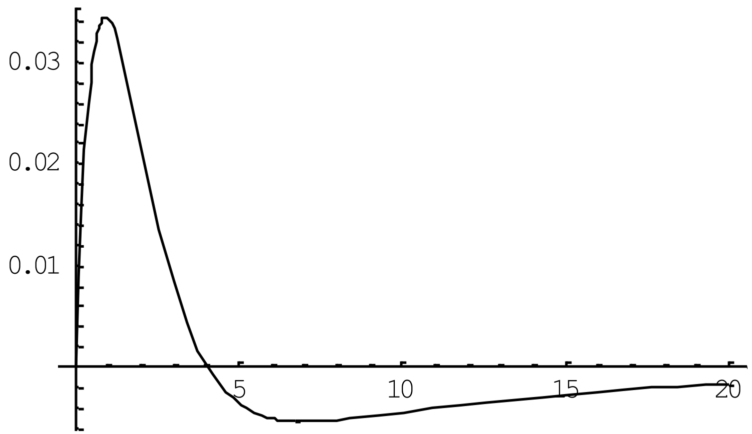
- By setting the time constant same as the previous stage T_1, i.e. $C_1R_1 = C_2R_2$, the pole of T_1 and zero of T_2 are canceled. This is called pole-zero cancellation (PZC).



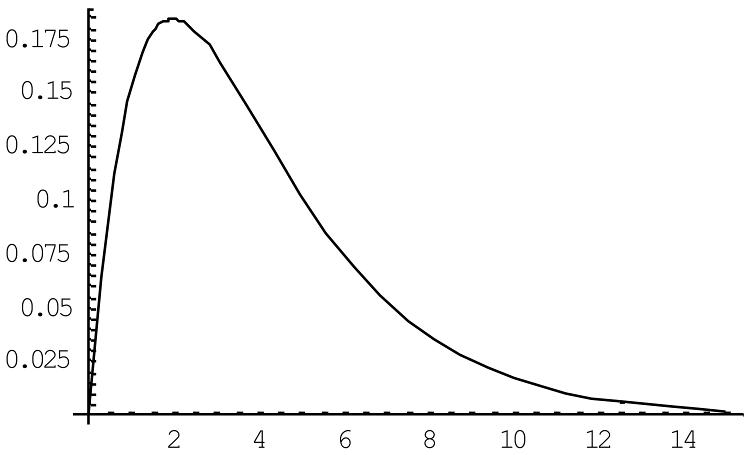
$$T_1(s) = -\frac{R_1}{1 + sC_1R_1} \qquad T_2(s) = \frac{R_3}{R_2 + R_3}\frac{1 + sC_2R_2}{1 + sC_2(R_2||R_3)}$$

Pole-Zero Cancellation (PZC)

- If pole and zero are not matched…

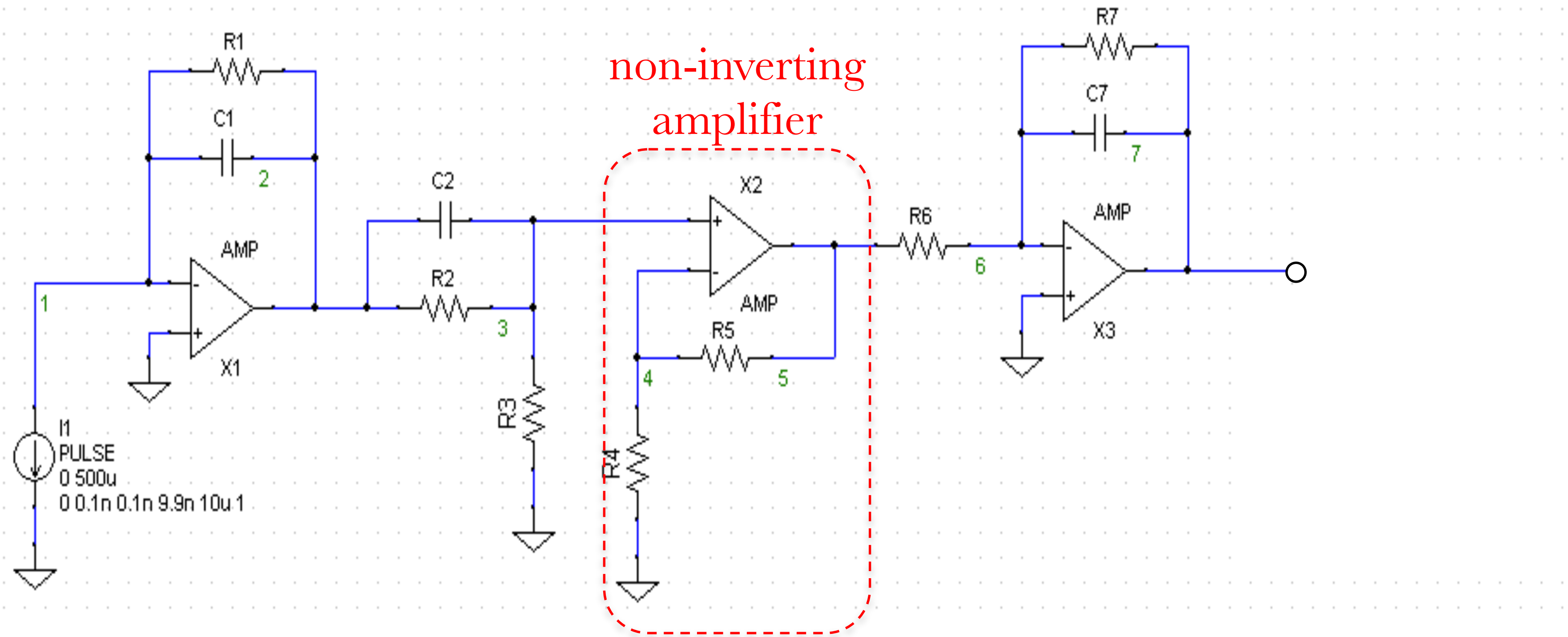


- Pole and zero are properly matched.



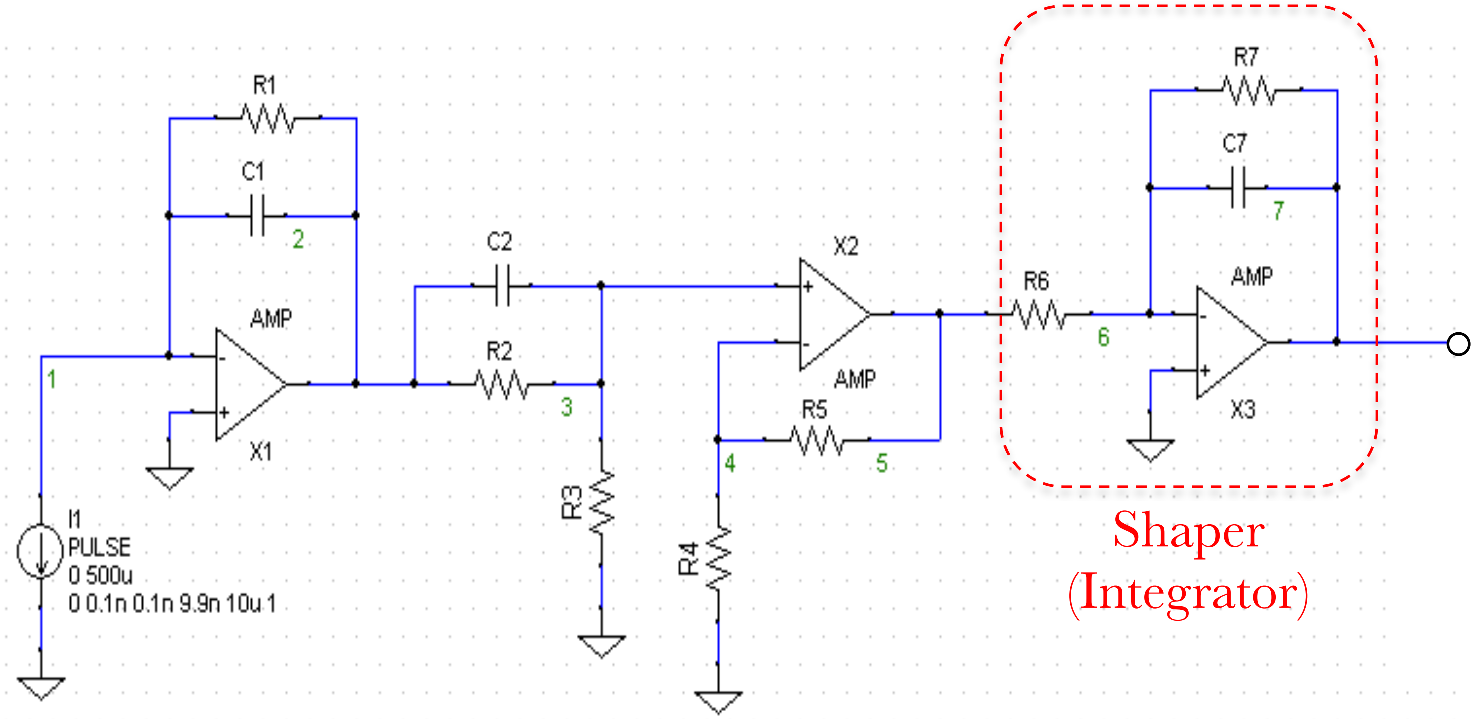
Non-Inverting Amplifier (already explained)

- DC attenuation by T_2 is recovered by setting $R_5/R_4 = R_2/R_3$.



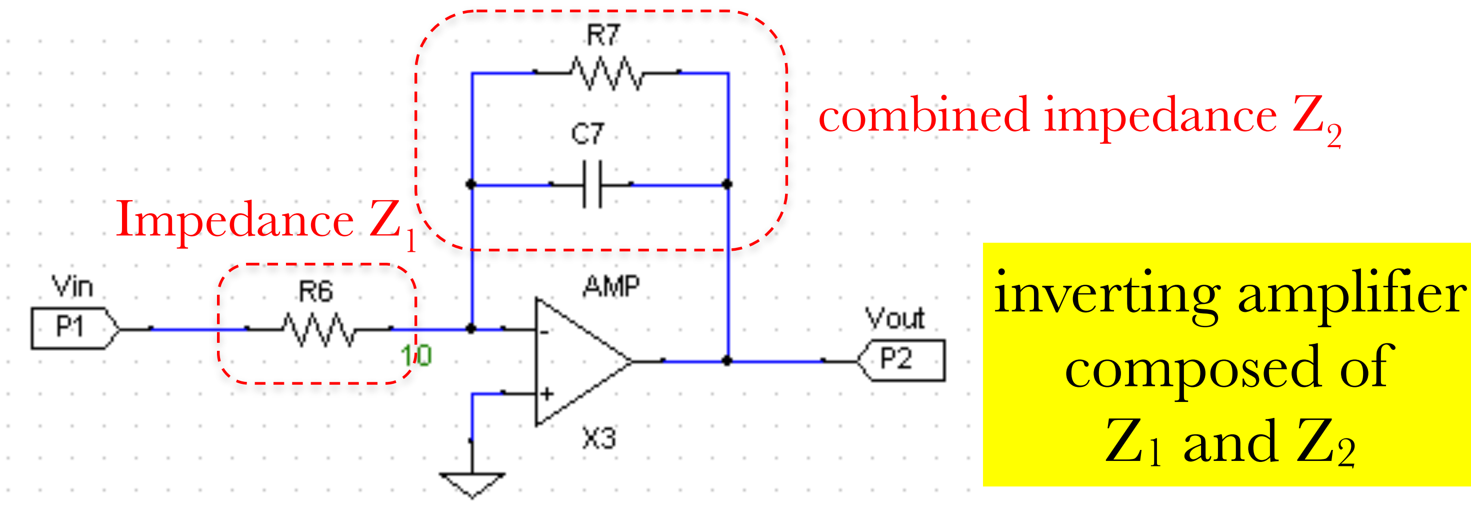
$$T_3(s) \equiv \frac{V_{out}(s)}{V_{in}(s)} = 1 + \frac{R_5}{R_4}$$

Shaping Amplifier (Shaper)



Shaping Amplifier (Shaper)

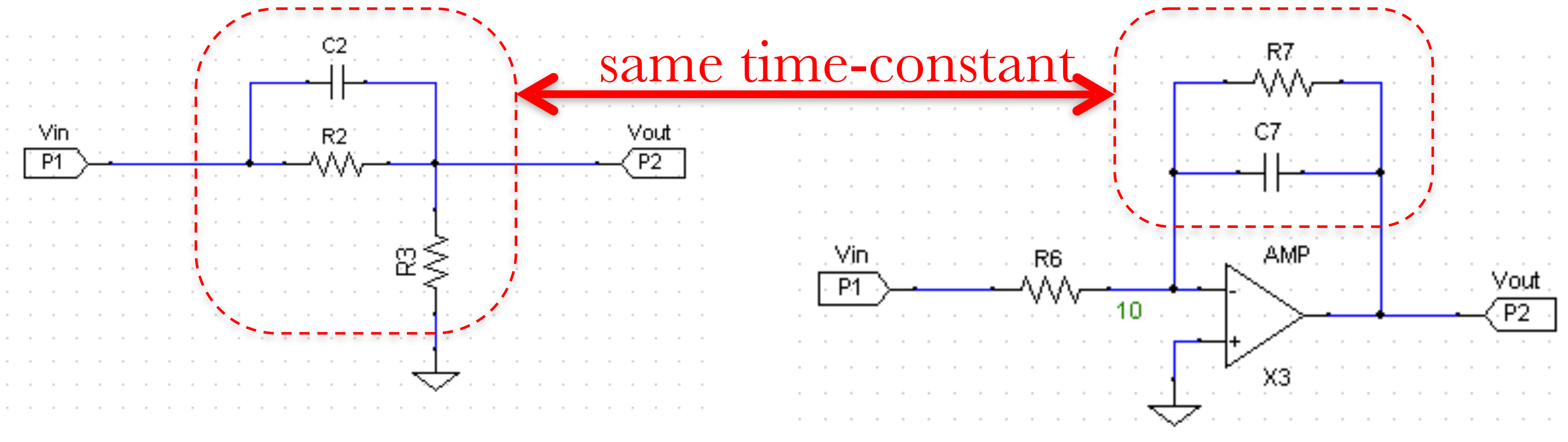
- Low-Pass Filter (LPF) property or integration property. The time constant C_7R_7 defines the shaping time.
- To make a "critical state," where the signal tail is just before vibration, equalize T_2 and T_4 poles by designing as $C_7R_7=C_2(R_2 \,||\, R_3)$.



$$T_4(s) \equiv \frac{V_{out}(s)}{V_{in}(s)} = -\frac{Z_2}{Z_1} = -\frac{R_7 || \frac{1}{sC_7}}{R_6} = -\frac{1}{R_6}\frac{1}{\frac{1}{R_7}+sC_7} = -\frac{R_7}{R_6}\frac{1}{1+sC_7R_7}$$

Shaping Amplifier (Shaper)

- Low-Pass Filter (LPF) property or integration property. The time constant C_7R_7 defines the shaping time.
- To make a "critical state," where the signal tail is just before vibration, equalize T_2 and T_4 poles by designing as $C_7R_7=C_2(R_2 \,||\, R_3)$.



$$T_2(s) = \frac{R_3}{R_2 + R_3} \frac{1 + sC_2R_2}{1 + sC_2(R_2||R_3)} \qquad T_4(s) = -\frac{R_7}{R_6} \frac{1}{1 + sC_7R_7}$$

pole

avoid vibrating state
by equalizing poles

pole

Synthesized Transfer Function

$$T(s) = I_0(s)T_1(s)T_2(s)T_3(s)T_4(s)$$

$$= Q\left(\frac{R_1}{1+sC_1R_1}\right)\left(\frac{R_3}{R_2+R_3}\frac{1+sC_2R_2}{1+sC_2\left(R_2 \parallel R_3\right)}\right)\left(1+\frac{R_5}{R_4}\right)\left(\frac{R_7}{R_6}\frac{1}{1+sC_7R_7}\right)$$

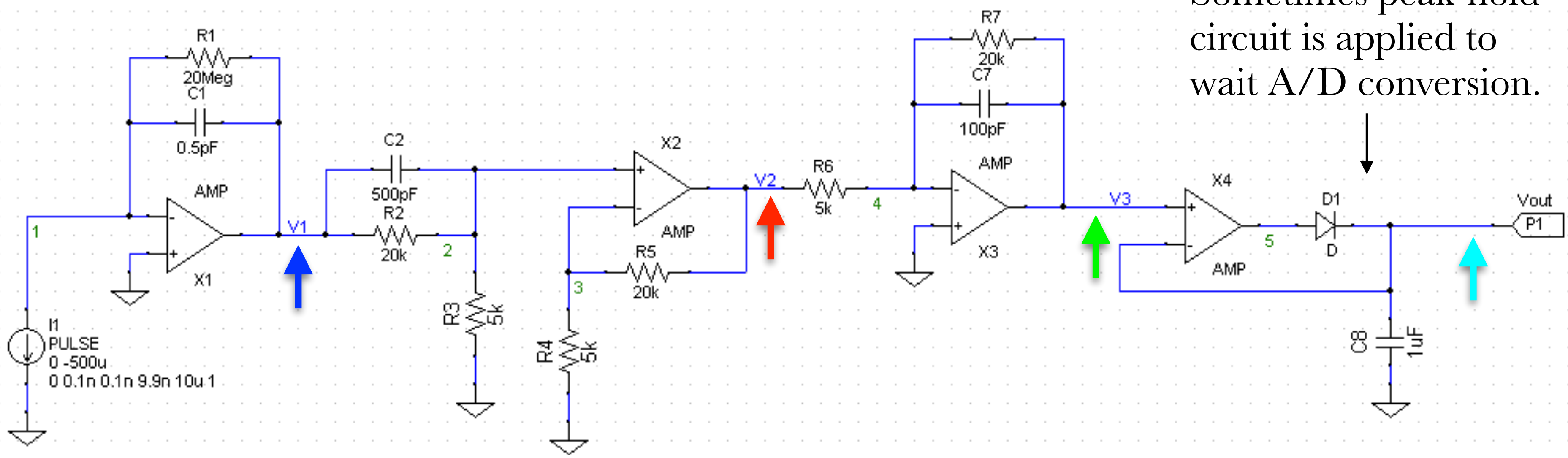
$$= QR_1\frac{R_7}{R_6}\frac{1}{(1+sC_7R_7)^2} \qquad \text{where } C_1R_1 = C_2R_2,\ \frac{R_2}{R_3} = \frac{R_5}{R_4},\ C_2\left(R_2 \parallel R_3\right) = C_7R_7$$

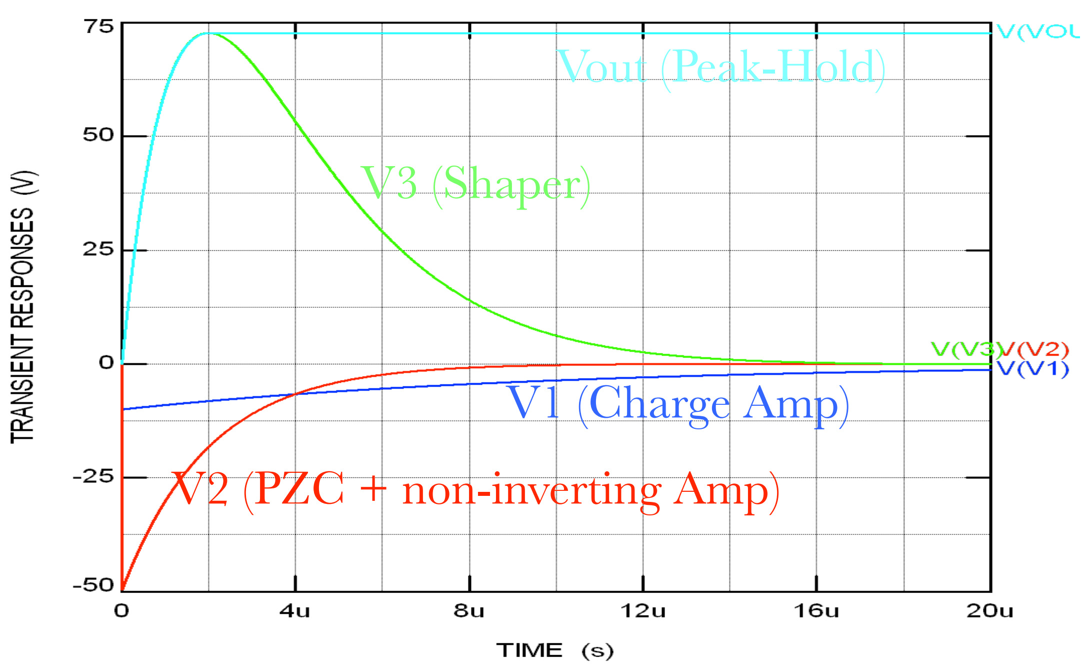
$$\Leftrightarrow \quad QR_1\frac{R_7}{R_6}\left(\frac{1}{C_7R_7}\right)^2 te^{-\frac{t}{C_7R_7}}$$

※ $te^{-at} \Leftrightarrow \dfrac{1}{(s+a)^2}$ is used here.

SPICE Simulation

Sometimes peak-hold circuit is applied to wait A/D conversion.



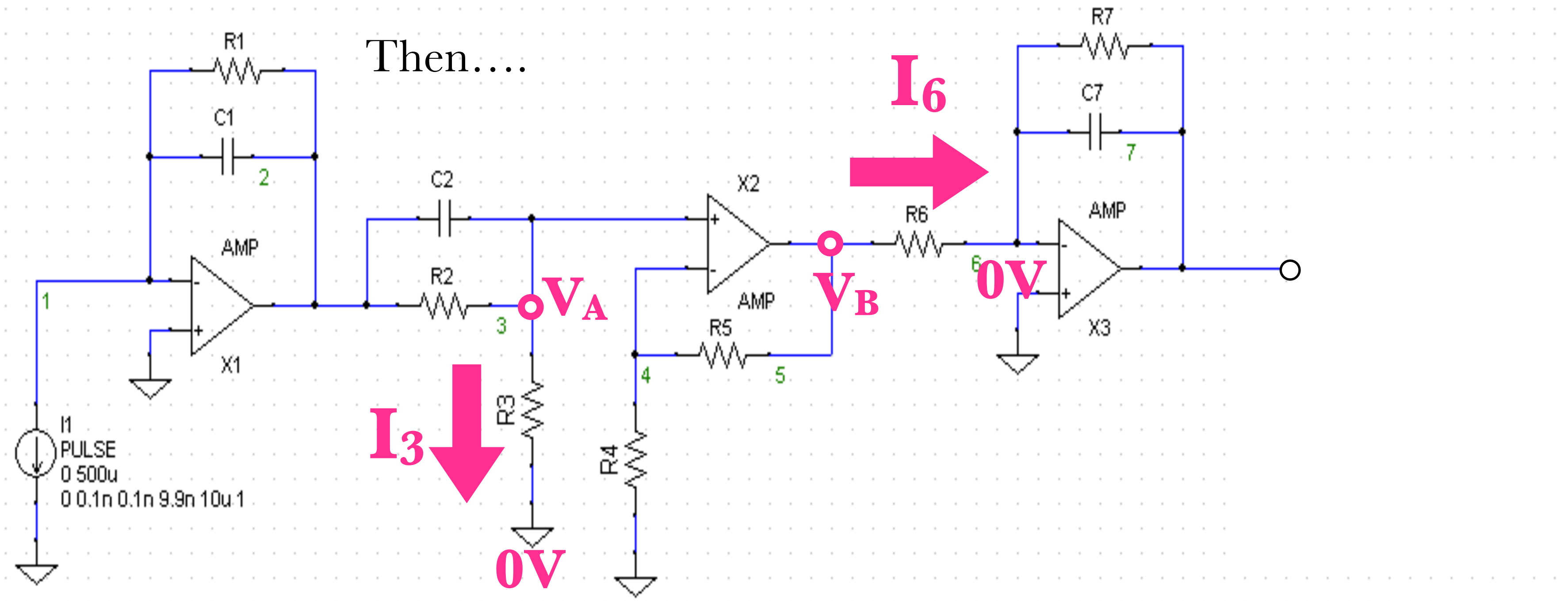


Consideration to minimize the circuit

Two currents should be completely proportional.

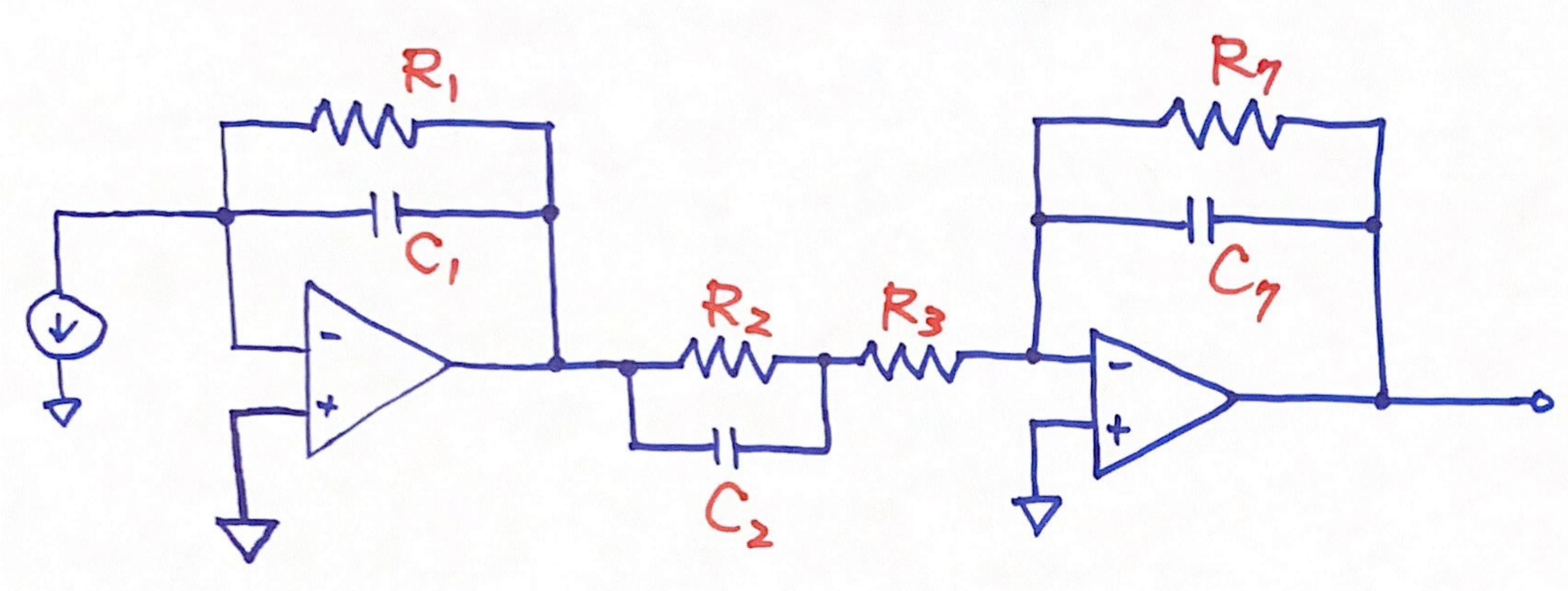
This is because V_A is proportional to V_B and
V_A and V_B are respectively proportional to I_3 and I_6
due to Ohom's law.

Then….



Consideration to minimize the circuit

The following circuit should work as same as the previous one, with only the amplification difference.



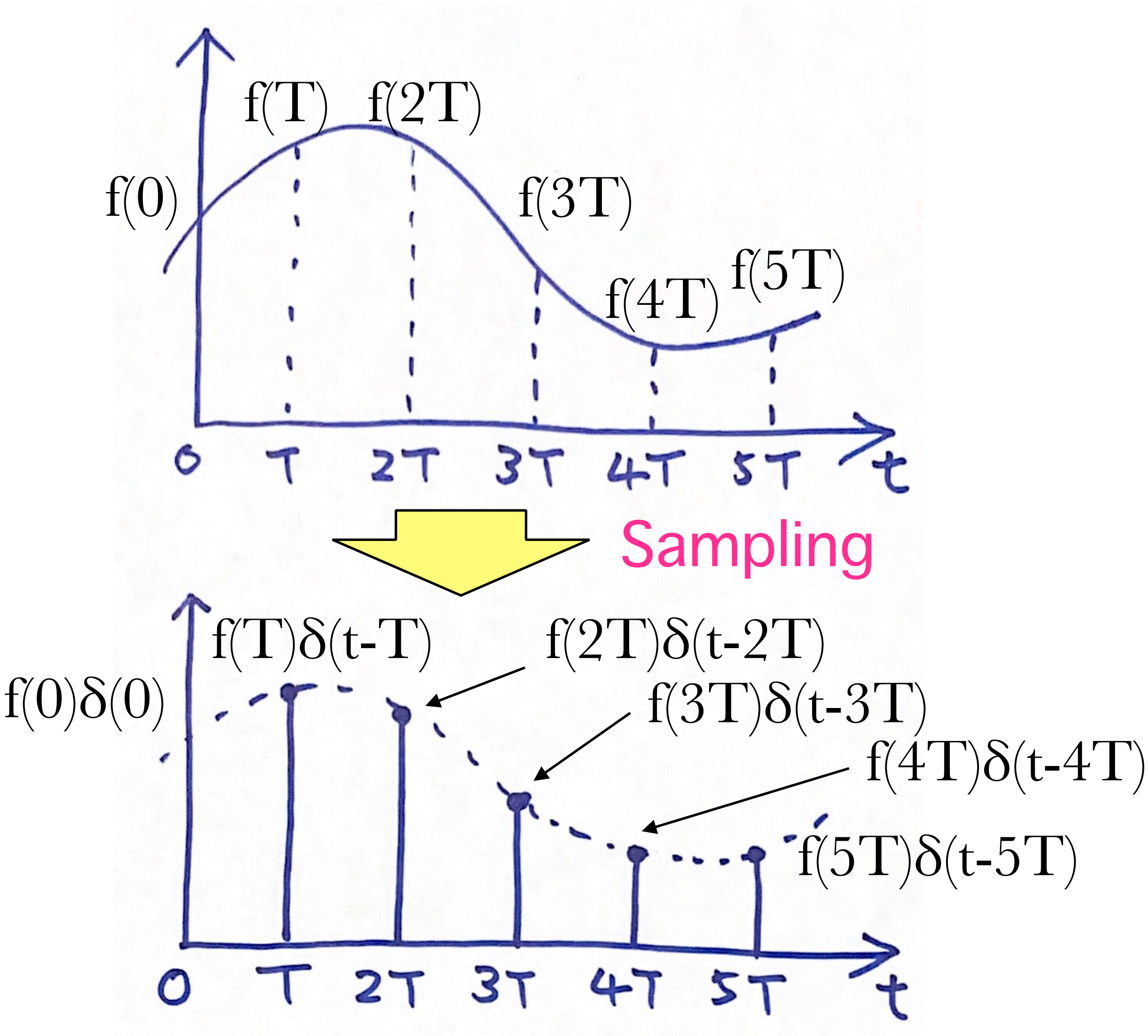
Group Homework Problem:

Please calculate the transfer function of this circuit to show that the above sentence is true.

You can also use the same capacitor and resister parameters as the original circuit to draw a graph that shows time dependence of the output voltage.

z-transform for digital filtering

Analog signal to digital signal



Laplace transform of the Sampled signal

When the signal f(t) is sampled with the period T,
the obtained sampled signal f*(t) is

$$f^*(t) = f(0)\delta(t) + f(T)\delta(t-T) + f(2T)\delta(t-2T) + f(3T)\delta(t-3T) + \cdots$$

If the signal is Laplace-transformed,

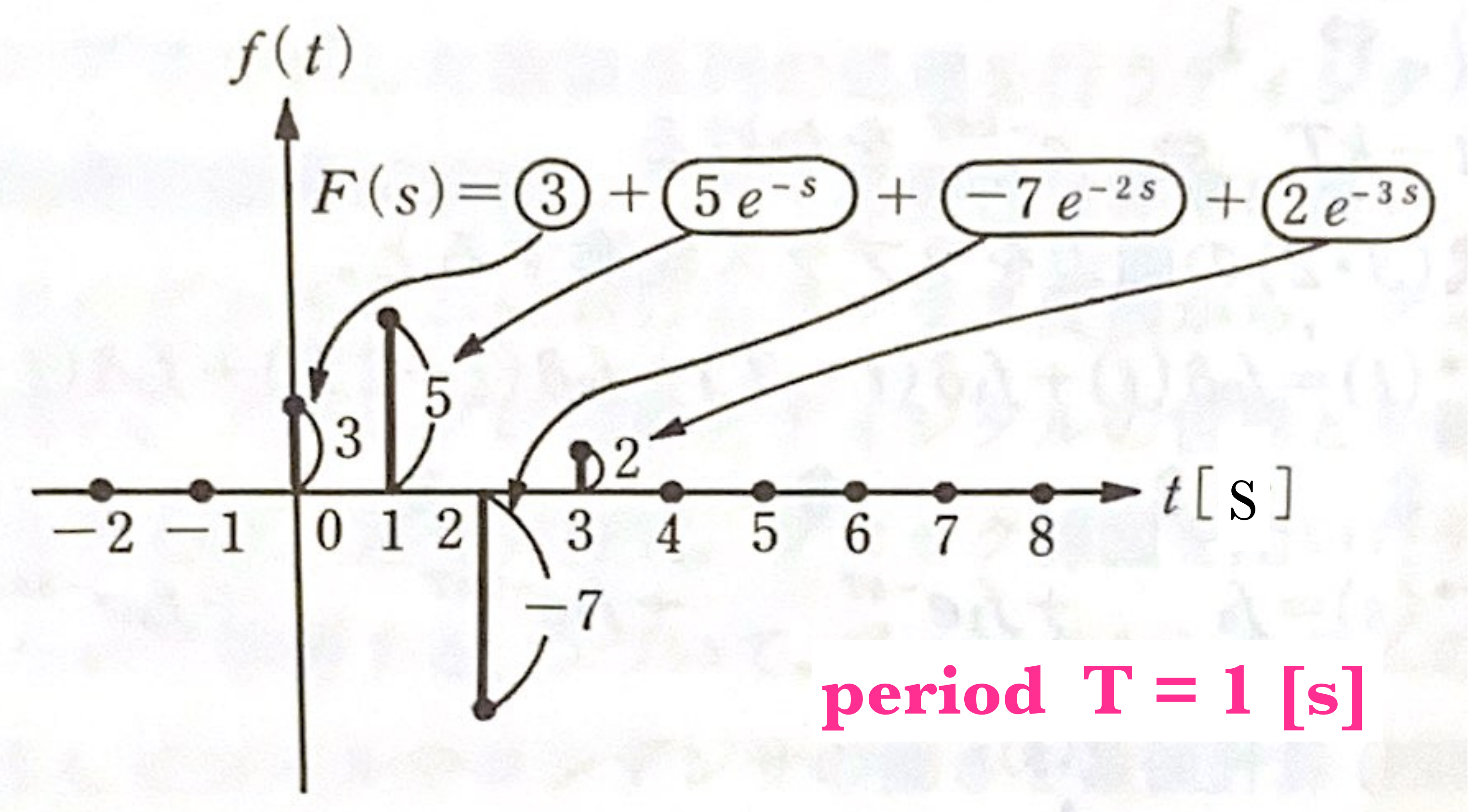
$$F^*(s) = f(0) + f(T)e^{-sT} + f(2T)e^{-2sT} + f(3T)e^{-3sT} + \cdots$$

because of the following Laplace transform pairs.

$$\delta(t) \quad \Leftrightarrow \quad 1$$

$$t \to t - kT \Leftrightarrow \text{multiplication of } e^{-ksT}$$

Example



z-transform

Look at the Laplace transform again.

$$F^*(s) = f(0) + f(T)e^{-sT} + f(2T)e^{-2sT} + f(3T)e^{-3sT} + \cdots$$

If we replace e^{-sT} by z^{-1},

$$F^*(s) = f(0)(e^{-1})^0 + f(1)(e^{-1})^1 + f(2)(e^{-1})^2 + f(3)(e^{-1})^3 + \cdots$$

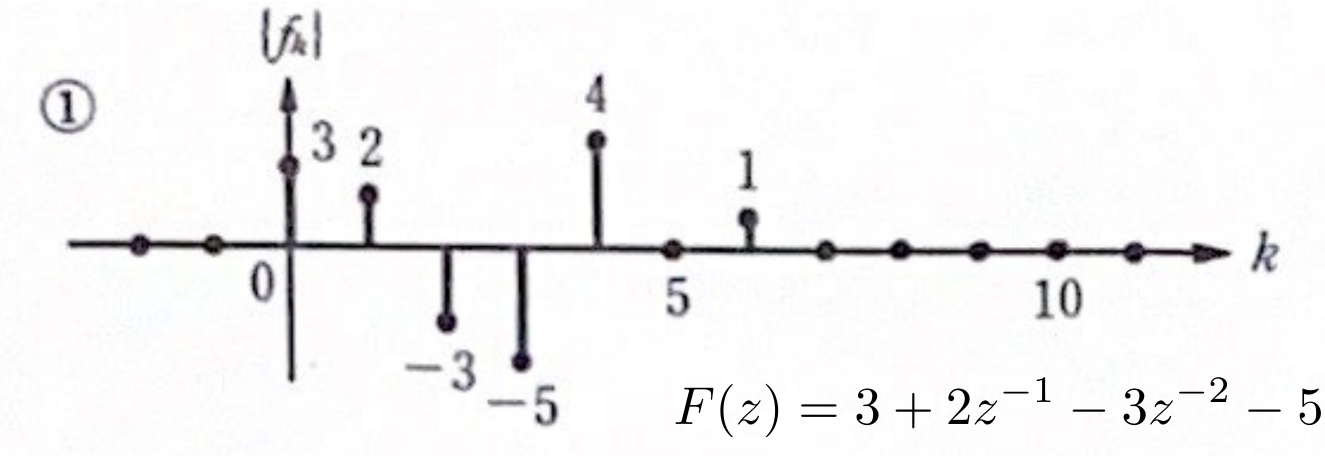
Here, z^{-1} denotes "one period (T) delay." Therefore,

$$z^{-1} \rightarrow T \text{ [s] delay}$$
$$z^{-2} = (z^{-1})^2 \rightarrow 2T \text{ [s] delay}$$
$$z^{-3} = (z^{-1})^3 \rightarrow 3T \text{ [s] delay}$$
$$z^{-4} = (z^{-1})^4 \rightarrow 4T \text{ [s] delay}$$

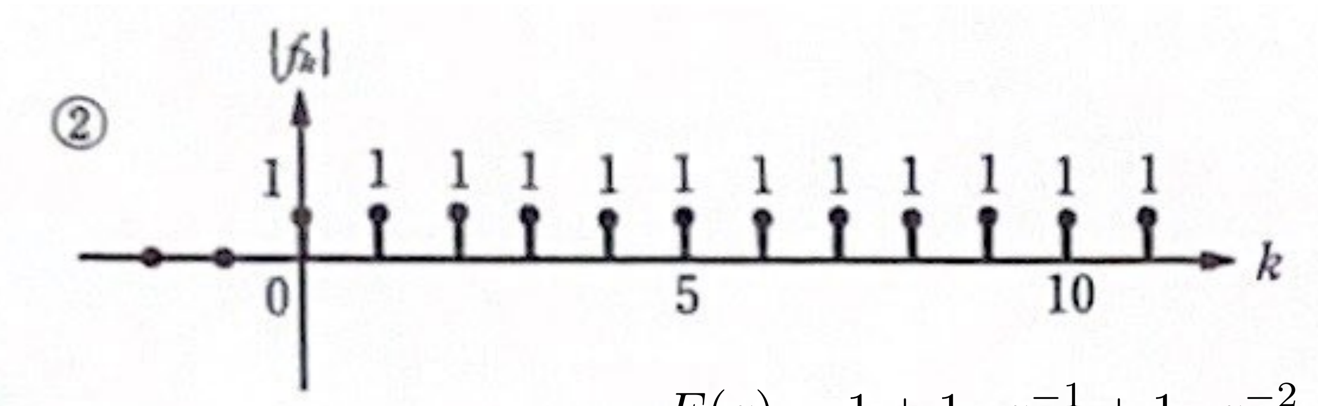
So, the discrete signal array { f(0), f(T), f(2T), f(3T), … } is transformed to

$$F(z) = \sum_{k=0}^{\infty} f(kT)(z^{-1})^k$$

z-transform Examples



$$F(z) = 3 + 2z^{-1} - 3z^{-2} - 5z^{-3} + 4z^{-4} + z^{-7}$$



$$F(z) = 1 + 1 \cdot z^{-1} + 1 \cdot z^{-2} + 1 \cdot z^{-3} + \cdots$$

$$= \frac{1}{1 - z^{-1}}$$

z-transform table

z-transform

f(kT)	F(z)
$\delta_k = \begin{cases} 1 & (k = 0) \\ 0 & (k \neq 0) \end{cases}$	1
$u_k = \begin{cases} 1 & (k \geq 0) \\ 0 & (k < 0) \end{cases}$	$\frac{1}{1 - z^{-1}}$
$e^{k\alpha T}$	$\frac{1}{1 - e^{\alpha T} z^{-1}}$

Laplace transform
(for comparison)

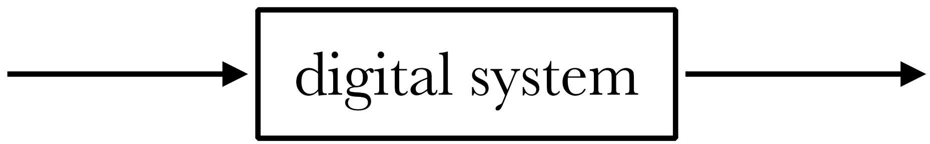
f(t)	F(s)
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$e^{\alpha t} u(t)$	$\frac{1}{s - \alpha}$

in/out correlation of digital system

Let me denote $x(kT)$, i.e. k-th data, as x_k

$$\left.\begin{aligned} x_0 &= 5 \\ x_1 &= 4 \\ x_2 &= -2 \\ x_3 &= 8 \\ x_4 &= 9 \\ &\vdots \end{aligned}\right\}$$

$(x_k = 0 \text{ for } k < 0)$



For example,
one sample period (T[s]) delay

$$\left\{\begin{aligned} y_0 &= 0 \\ y_1 &= 5 \\ y_2 &= 4 \\ y_3 &= -2 \\ y_4 &= 8 \\ &\vdots \end{aligned}\right.$$

$$y_k = x_{k-1}$$

← "difference equation"

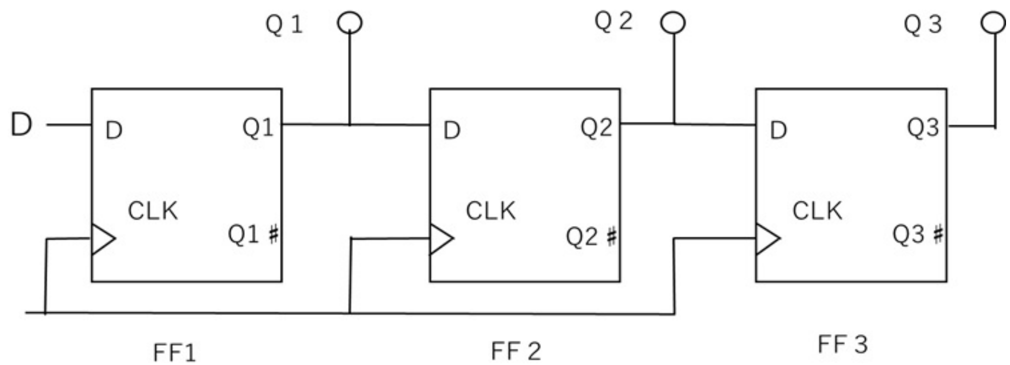
$Y(z)$, z-transform of y_k, is calculated as

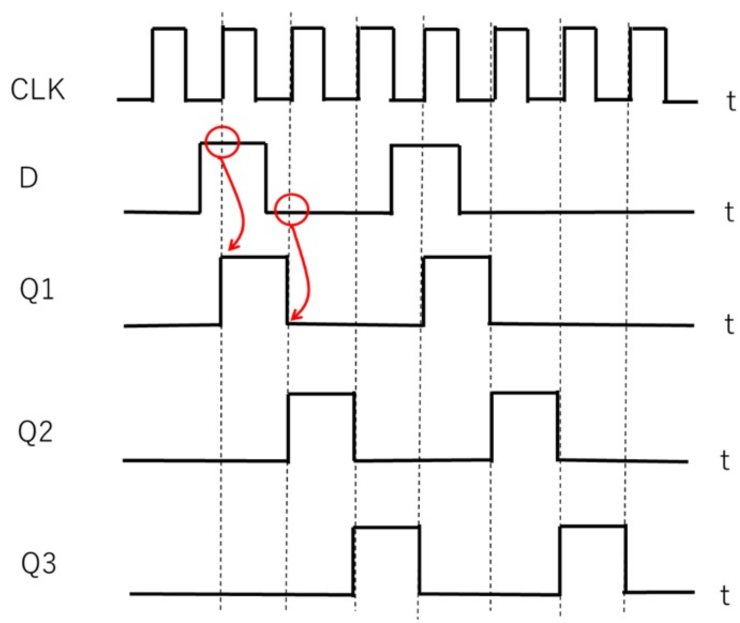
$$\begin{aligned} Y(z) &= y_0 + y_1 z^{-1} + y_2 z^{-2} + y_3 z^{-3} + \cdots \\ &= x_{-1} + x_0 z^{-1} + x_1 z^{-2} + x_2 z^{-3} + \cdots \\ &= z^{-1}(x_0 + x_1 z^{-1} + x_2 z^{-2} + \cdots) \quad = z^{-1} \sum_{k=0}^{\infty} x_k z^{-k} \\ &= z^{-1} X(z) \end{aligned}$$

multiplication of z^{-1} works as one sample delay!

One-sample delay circuit

In digital circuits, one-sample delay is established by a D-type flip-flop (DFF). A cascade of DFF is called a shift-register.

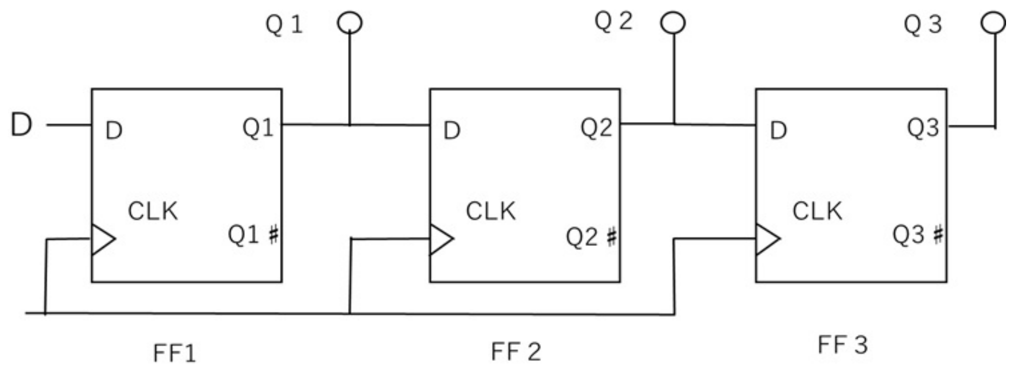




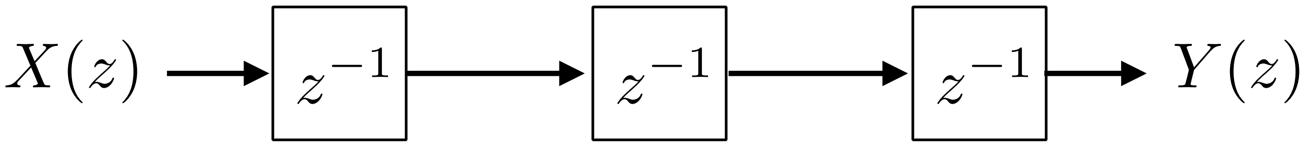
Figures from
https://bit.ly/3KvbCq2

One-sample delay circuit

In digital circuits, one-sample delay is established by a D-type flip-flop (DFF). A cascade of DFF is called a shift-register.



z-transform of the shift-registe



$$Y(z) = z^{-3}X(z)$$

Difference equation

The operation of the filter is described by a difference equation that relates y_k as a function of the present input sample x_k and any number of past input and output samples

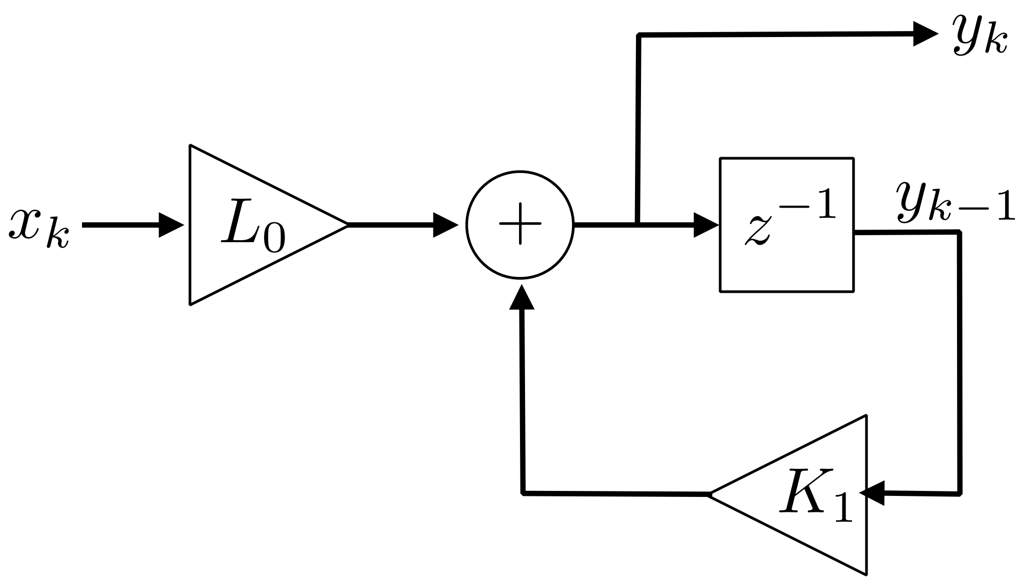
Recursion formula

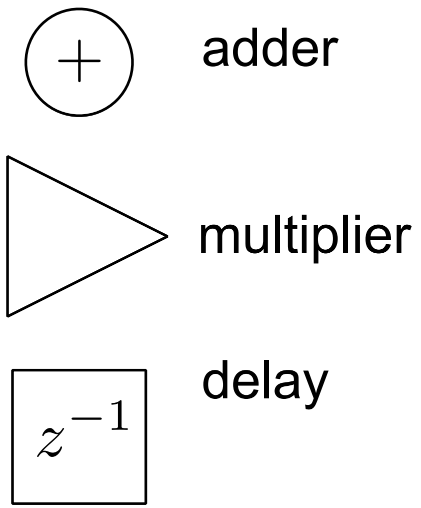
$$y_k = \sum_{i=0}^{q} L_i x_{k-i} + \sum_{i=1}^{m} K_i y_{k-i}$$

A simple 1st-order Difference equation

Recursion formula

$$y_k = L_0 x_k + K_1 y_{k-1}$$





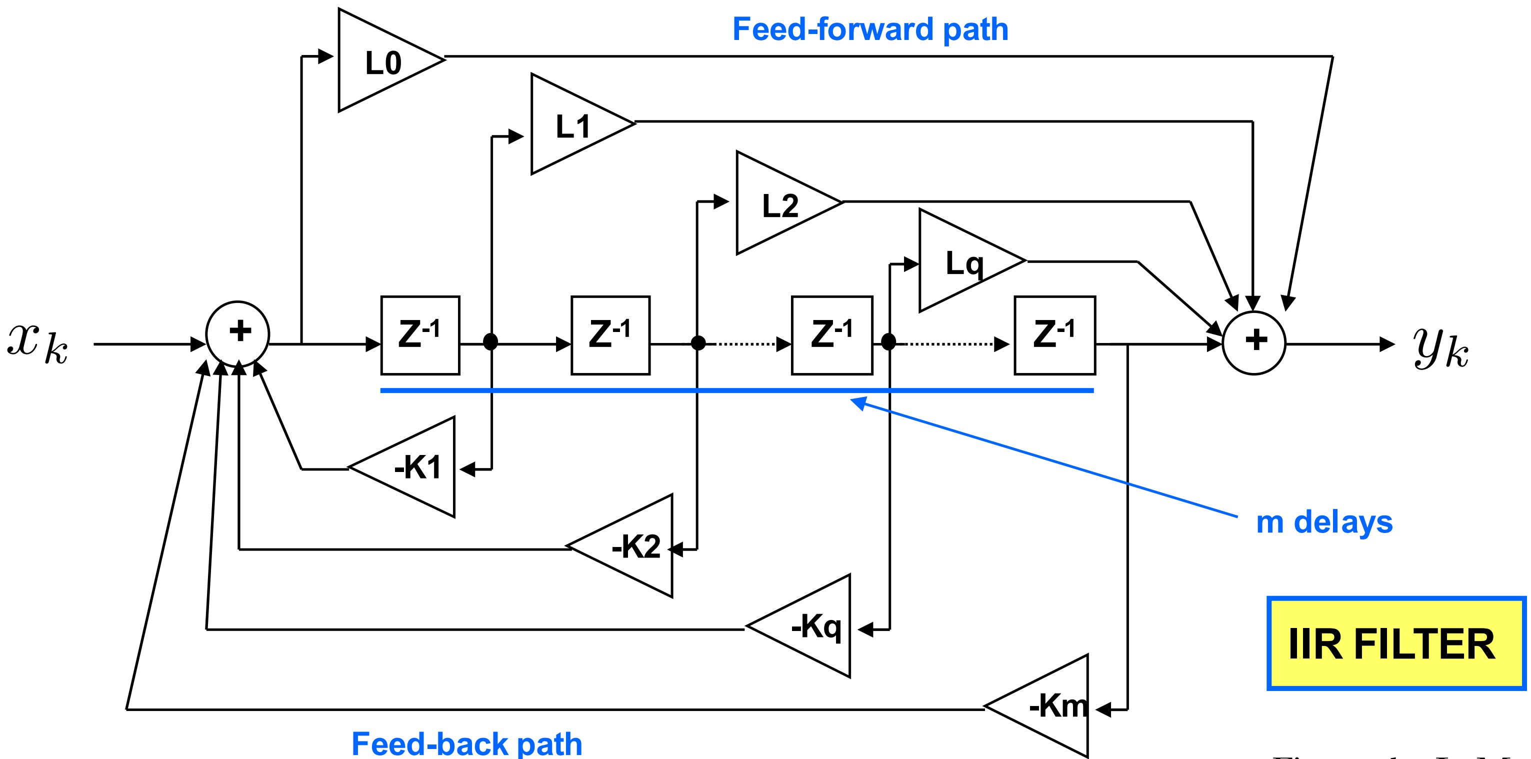
Digital Networks

Difference equation

$$y_k = \sum_{i=0}^{q} L_i x_{k-i} + \sum_{i=1}^{m} K_i y_{k-i}$$

System function

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{q} L_i z^{-i}}{1 + \sum_{i=1}^{m} K_i z^{-i}}$$



Figures by L. Musa @ CERN

Digital Networks

System function

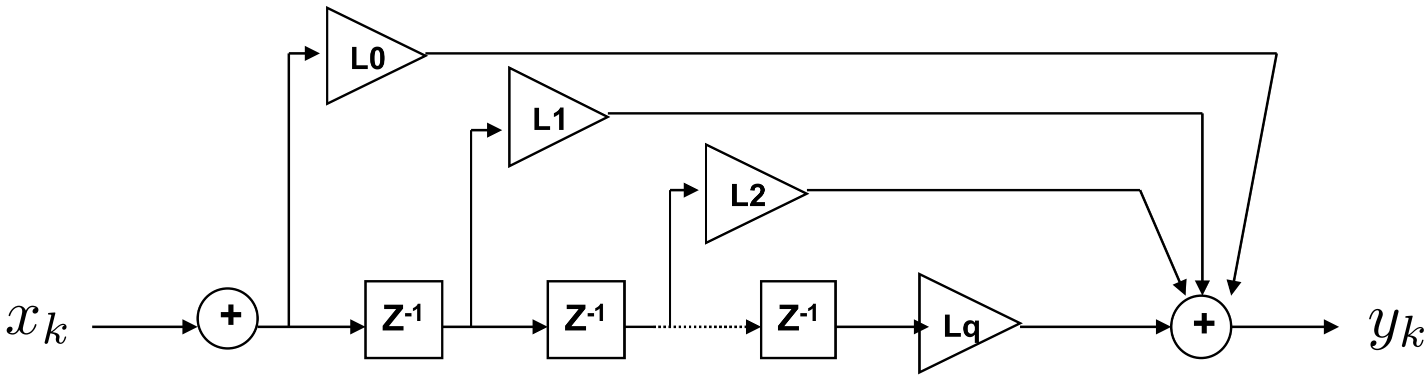
$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{q} L_i z^{-i}}{1 + \sum_{i=1}^{m} K_i z^{-i}}$$

Difference equation

$$y_k = \sum_{i=0}^{q} L_i x_{k-i} + \sum_{i=1}^{m} K_i y_{k-i}$$

If all Ki's are 0

Non-recursive

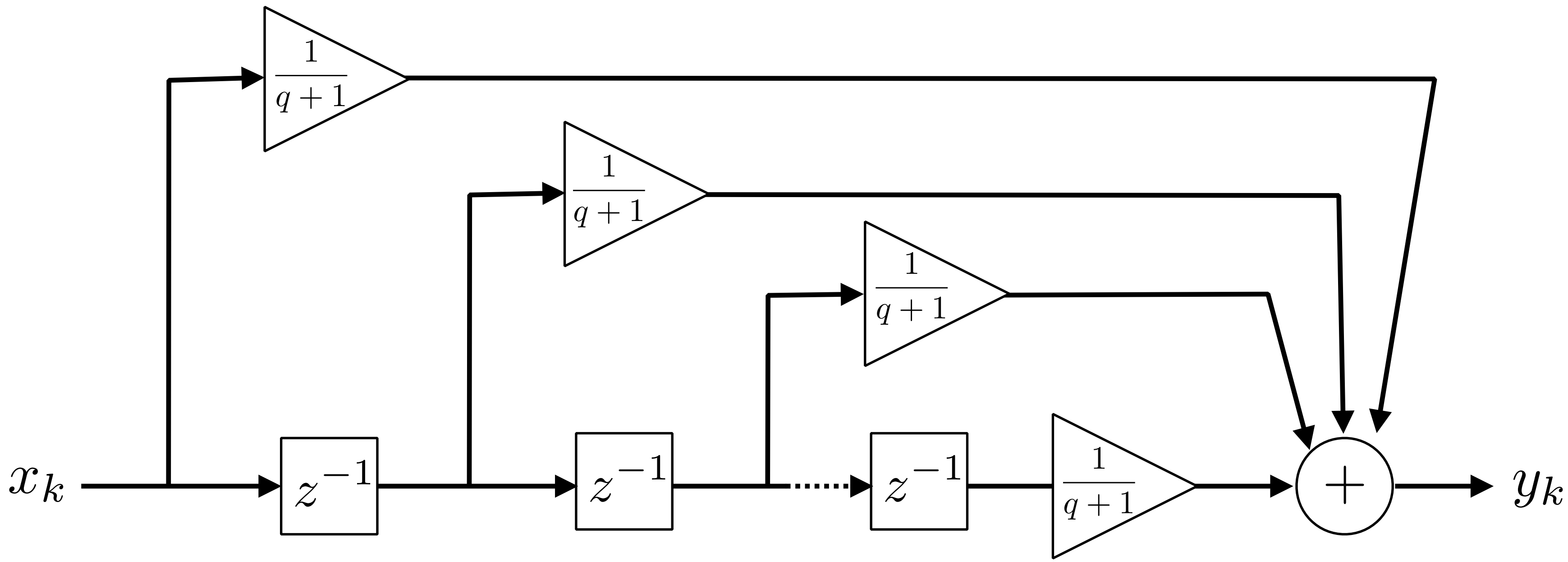


q delays

FIR FILTER

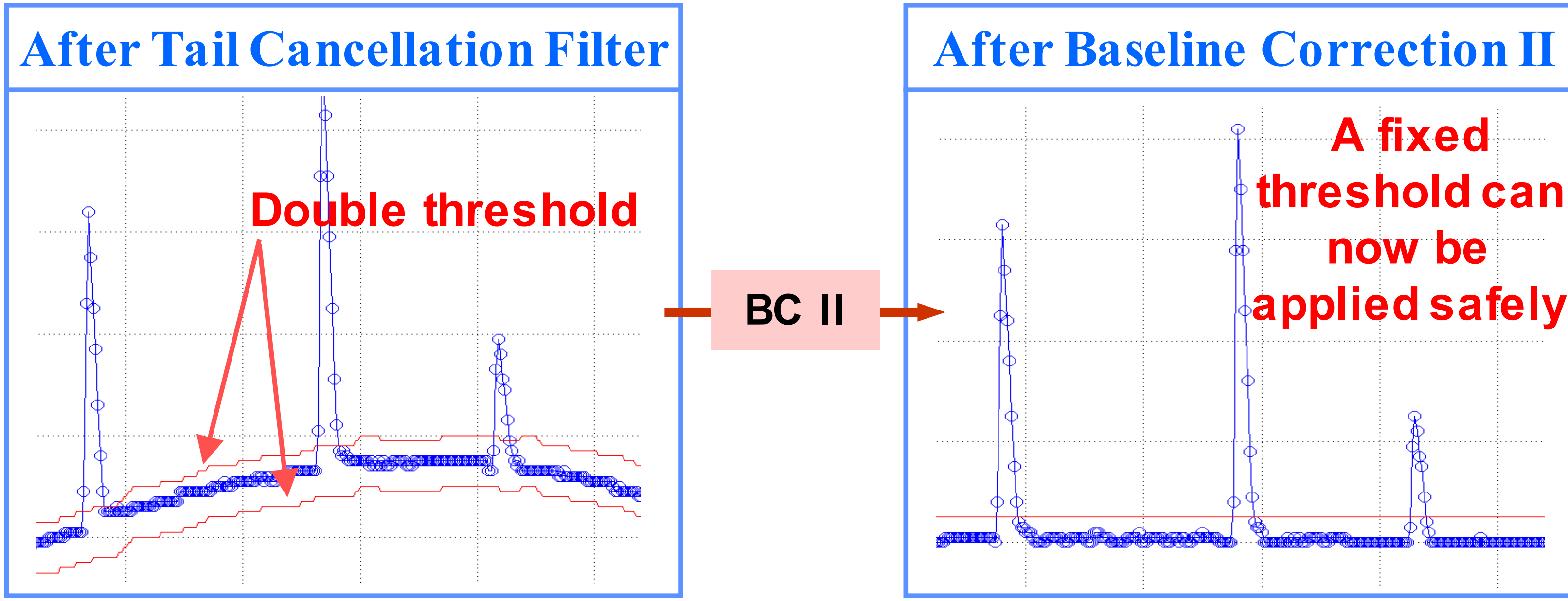
Figures by L. Musa @ CERN

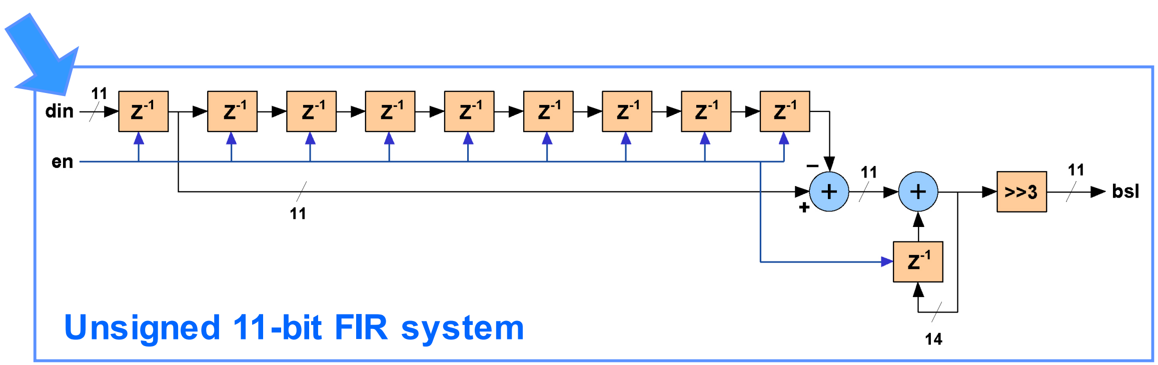
Example of FIR filter: Moving Average Filter (MAF)



$$H(z) = \frac{1}{q+1} \sum_{i=0}^{q} z^{-i}$$

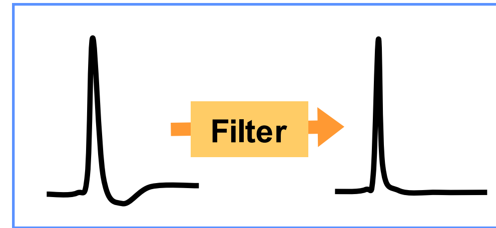
Baseline correction using MVA





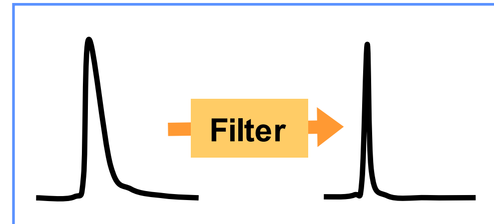
Figures by L. Musa @ CERN

- **Functions**
  - **signal (ion) tail suppression**
  - **pulse narrowing ⇒ improves cluster separation**
  - **gain equalization**

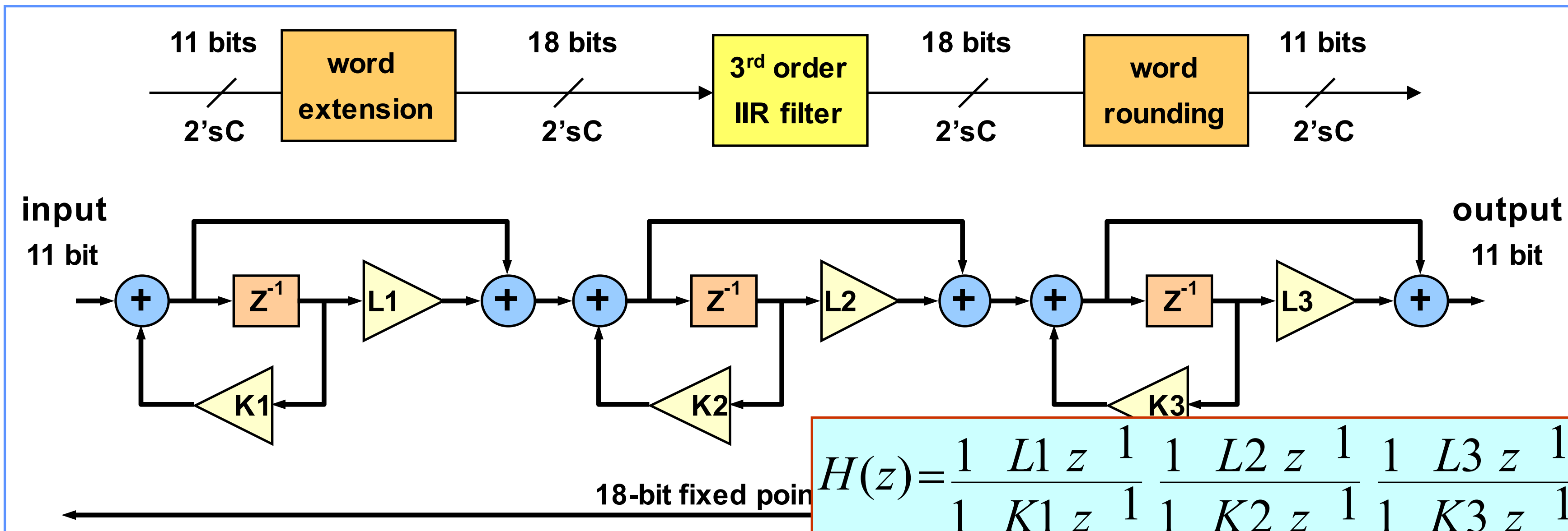


compensates undershoot

- **Architecture**
  - **3rd order IIR filter**
  - **18-bit fixed point 2'sC arithmetic**
  - **single channel configuration ⇒ 6 coefficients / channel**



Narrows the pulse



$$H(z)=\frac{1\quad L1\ z\quad 1}{1\quad K1\ z\quad 1}\ \frac{1\quad L2\ z\quad 1}{1\quad K2\ z\quad 1}\ \frac{1\quad L3\ z\quad 1}{1\quad K3\ z\quad 1}$$

Summary

- An example of frontend analog circuits composed of OP amps was shown. Its analysis method was presented using the Laplace transform.
- z-transform was explained, which is useful for analysis of digital circuits. FIR and IIR filters were shown.
- Homework problem : see pages 34 and 35.

fin

backup slides

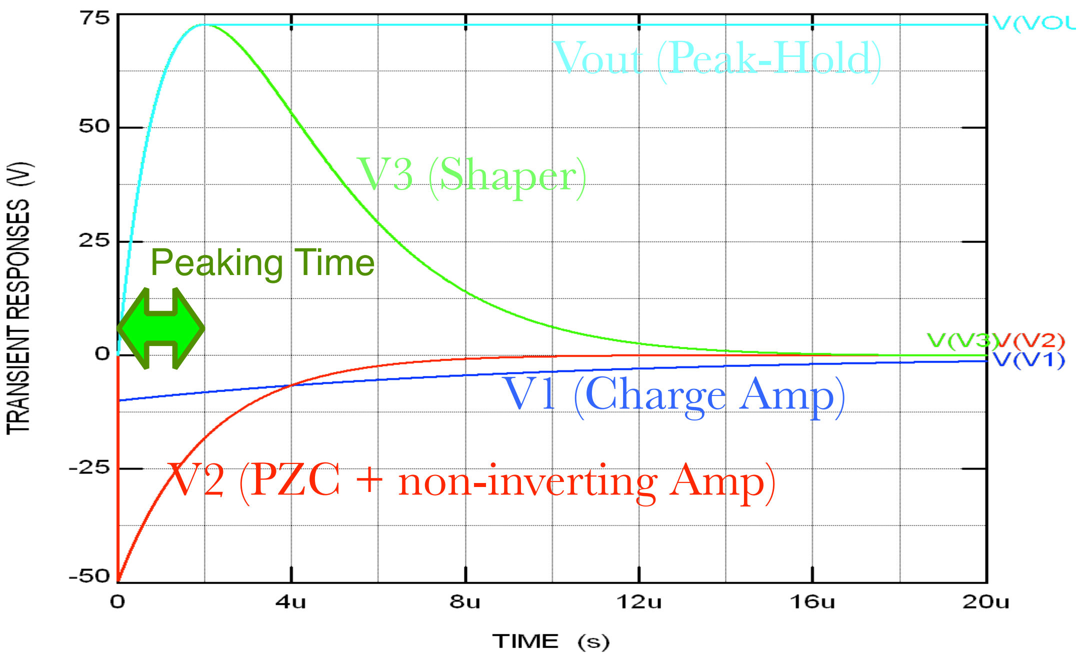
Parameters

Peak $v_{max} = QR_1 \frac{R_7}{R_6} \frac{1}{C_7 R_7} \frac{1}{e}$ is obtained at $t = C_7 R_7$, which is called “peaking time.”

Shaping time is $\sim 1.04\, C_7 R_7$.

Integral of the pulse gives $\int_0^\infty v(t)dt = QR_1 \frac{R_7}{R_6}$.

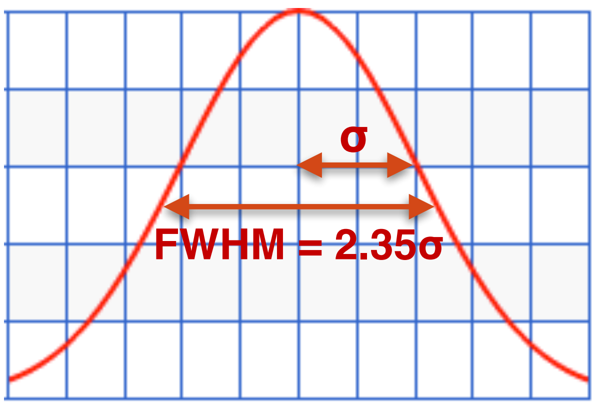
Peak and Integral are both proportional to the injection charge Q.



Definition of Shaping Time

Shaping time is defined as the σ of gaussian-shape pulse. FWHM is known to be $2\sqrt{2\ln 2} \approx 2.35$ times larger than σ.

For non-gaussian pulse, therefore, FWHM/2.35 can be used as shaping time instead.



$$\mathrm{FWHM} = 2.45\ C_7 R_7$$

$$\therefore \mathrm{ST} = 1.04\ C_7 R_7$$

※ ST/PT depends on shaping type.