Scaling symmetry and the Smarr relation

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Smarr relation and Thermodynamics

- The first law

$$\delta M = T_H \delta S - \Phi \delta Q - \Omega \delta J \tag{1}$$

- The Smarr relation (3 dim'nal) [Smarr '73]

$$M = \frac{1}{2} T_H S - \frac{1}{2} \Phi Q - 2\Omega J \tag{2}$$

- Scaling symmetry gives Smarr relation
  - 3 dim Einstein gravity with an minimally coupled scalar hair in asymptotic AdS geometry. [Banados, Theisen '05]

Introduction

Scalar hairy Black holes

- No-hair theorem in 4 dim'nal flat space
- AAdS BH can have scalar hair with negative mass!
- AdS space

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\Sigma^2_{D-2} \quad (3)$$

- Lifshitz space

$$ds^2 = -r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 d\Sigma^2_{D-2} \quad (4)$$

  - anisotropic scaling of time and space

$$t \to \lambda^z t\,, \quad x \to \lambda x\,. \quad (5)$$

Introduction

1. 3 dimensional black holes
   - New Massive Gravity with non-minimally coupled scalar model
2. Higher dimensional black branes
   - Einstein-Maxwell-dilaton model (with scalar hair)

Scalar hairy Lifshitz BH in 3 dim'nal NMG

- Model : NMG coupled with a scalar field

$$I[g,\varphi] = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\Big[\eta\Big(\mathcal{L}_{EH} + \frac{1}{m^2}\mathcal{L}_K\Big) + \mathcal{L}_\varphi\Big]\,, \qquad (6)$$

$$\mathcal{L}_{EH} = R - 2\Lambda\,,$$

$$\mathcal{L}_K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\,,$$

$$\mathcal{L}_\varphi = -\frac{1}{2}(\partial\varphi)^2 - \frac{\alpha}{2}R\varphi^2 - V(\varphi)\,.$$

which admits the Lifshitz space

$$ds^2 = -r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 d\theta^2\,, \qquad (7)$$

$$\Lambda = -\frac{1}{2}(z^2+z+1)\,, \quad m^2 = \frac{1}{2}(z^2-3z+1)\,.$$

Reduced action

- Metric ansatz : $ds^2 = -e^{2A(r)} f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2$.
- Scalar hair : $\varphi = \varphi(r)$.
- Reduced action

$$I_{red}[A, f, \varphi, \lambda, Z] = \frac{1}{16\pi G} \int d^3x \; L_{\varphi NMG} \,, \tag{8}$$

where

$$\begin{aligned} L_{\varphi NMG} = -e^A \Big[\eta \Big(2r\Lambda + f' - 2f\lambda' - f'\lambda \Big) + \frac{1}{2} r f \varphi'^2 + rV(\varphi) \\ - \frac{\alpha}{2} e^{-A} (Z' + e^A f') \varphi^2 \Big] - \eta \Big(Z + \frac{2}{m^2} e^A f \lambda \Big)' \\ + \frac{\eta e^{-A}}{8m^2 r} \Big[8e^A \lambda Z + (Z' - e^A f')^2 - \frac{4}{r} Z Z' + \frac{4}{r^2} Z^2 \Big] . \end{aligned}$$

Scaling symmetry

- Scaling transformation

$$\delta_\sigma I_{red} = \frac{1}{16\pi G}\int dt d\theta dr S' \,, \tag{9}$$

$$\delta_\sigma f = \sigma(2f - rf')\,,\;\; \delta_\sigma e^A = \sigma(-2 - rA')e^A\,,\;\; \delta_\sigma \varphi = -\sigma r \varphi'\,.$$

- Noether charge, $C = C(r)$

$$8G\,C \equiv \Theta(\delta_\sigma f) + \Theta(\delta_\sigma Z) + \Theta(\delta_\sigma \lambda) + \Theta(\delta_\sigma \varphi) - S \tag{10}$$

$$8G\,\eta\, C(r) = e^A\Big[-1 + \lambda - \frac{e^{-A}}{4m^2 r}(Z' - e^A f') + \eta\frac{\alpha}{2}\varphi^2\Big](2f - rf') + \lambda Z + e^A\Big[-2rf\lambda' + \eta\, r^2 f \varphi'^2\Big] + \frac{e^{-A}}{2m^2r^2}Z(2Z - rZ')$$

which is indep. of the scalar potential V.

Scaling symmetry

Meaning of C?

$$C = \frac{\eta}{8G}\Big[e^{A}\Big(-1+\lambda-\frac{e^{-A}}{4m^2 r}(Z'-e^{A}f')+\eta\frac{\alpha}{2}\varphi^2\Big)(2f-rf') + \lambda Z + e^{A}\Big(-2rf\lambda'+\eta r^2 f\varphi'^2\Big)+\frac{e^{-A}}{2m^2r^2}Z(2Z-rZ')\Big] \tag{11}$$

⇒ Compare with physical quantities of black holes

At the horizon

Bekenstein-Hawking-Wald entropy

- direct computation

$$\frac{\kappa}{2\pi}\mathcal{S}_{BHW} = \frac{1}{16\pi G}\int_{\mathcal{H}} dx_{\mu\nu}\Delta K^{\mu\nu}(\xi_H) \tag{12}$$

$$K^{rt}(\xi_H) = \left[\eta\left(Z + 2e^{A}f\lambda - \frac{e^{-A}}{4m^2 r}Z(Z' - e^{A}f') + \frac{e^{-A}}{2m^2r^2}Z^2\right) - \frac{\alpha}{2}Z\varphi^2\right]_{r=r_H}$$

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi}e^{A(r_H)}f'(r_H) \tag{13}$$

- from $C = C(r_H)$

$$\Rightarrow T_H\mathcal{S}_{BHW} = C(r_H) \tag{14}$$

At the asymptotic infinity

Mass of black hole

- direct computation

$$\delta M_{ADT} = \frac{1}{16\pi G}\int_{r\to\infty} dx_{\mu\nu}\Big(\delta K^{\mu\nu}(\xi_T) - 2\xi_T^{[\mu}\Theta^{\nu]}(\delta\Psi)\Big) \qquad (15)$$

- 'on-shell' scaling transformation to get finite expression

$$f(r) = r^2 + \cdots\,, \quad e^{A(r)} = r^{z-1} + \cdots\,, \quad \varphi(r) = \varphi_\infty + \cdots$$

$$\hat{\delta}_\sigma f = \sigma(2f - rf')\,, \;\; \hat{\delta}_\sigma e^A = \sigma(z - 1 - rA')e^A\,, \;\; \hat{\delta}_\sigma \varphi = -\sigma r\varphi'$$

$$\therefore \hat{\delta}_\sigma M = \sigma(1+z)M \qquad (16)$$

- from $C = C(r\to\infty)$

$$\Rightarrow \sigma C(r\to\infty) = \hat{\delta}_\sigma M = \sigma(1+z)M \qquad (17)$$

The Smarr relation

Smarr relation of hairy Lifshitz black holes

$$(1+z)M = C = T_H \mathcal{S}_{BHW} \tag{18}$$

which is checked for several explicit examples using C.

- z=1, BTZ $(e^A = 1\,, f = r^2 - a) : C = \frac{\eta a}{4G}\left(1 + \frac{1}{2m^2}\right) = 2M$
- z=1, new type $(e^A = 1\,, f = r^2 - br + c) : C = \frac{\eta}{8G}\left(b^2 - 4c\right) = 2M$
- z=1, scalar hairy : $C = \frac{3}{4G}B^2(1+\nu) = 2M$
  [Henneaux, Martinez, Troncoso, Zanelli '04]
- z=3, $e^A = r^{z-1}\,, f = r^2 - a : C = -\frac{\eta a^2}{G} = (1+z)M$
  [Ayon-Beato, Garbarz, Giribet, Hassaine '09]
- scalar hairy Lifshitz BH : $C = -\frac{\eta}{8G}\frac{a^2(z+1)^3(3z-5)}{16(z-1)(z^2-3z+1)} = (1+z)M$
  [Ayon-Beato, Bravo-Gaete, Correa, Hassaine, Juarez-Aubry, Oliva '15]

Lifshitz Black Brane in higher dim

- Model : Einstein-Maxwell-dilaton gravity in D dim

$$I[g,\phi,\mathcal{A}] = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\Big(\mathcal{L}_{EH}+\mathcal{L}_{\phi\mathcal{A}}\Big)\,, \tag{19}$$

$$\mathcal{L}_{\phi\mathcal{A}} \equiv -\frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{\lambda\phi}\mathcal{F}^2\,, \tag{20}$$

$$\lambda = -\sqrt{2\frac{D-2}{z-1}}\,, \qquad \Lambda = -\frac{(D+z-2)(D+z-3)}{2}\,.$$

which admits the Lifshitz black brane with

$$e^{\phi} = \mu\, r^{\sqrt{2(D-2)(z-1)}}\left[1+\mathcal{O}\Big(\frac{1}{r}\Big)\right]\,, \tag{21}$$

$$\mathcal{A} = \sqrt{\frac{2(z-1)}{D+z-2}}\,\mu^{-\frac{\lambda}{2}}\, r^{D+z-2}\left[1+\mathcal{O}\Big(\frac{1}{r}\Big)\right]\, dt\,, \qquad \mathcal{F} = d\mathcal{A}\,. \tag{22}$$

Reduced action

- Metric ansatz : $ds^2 = -e^{2A(r)} f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma^2_{D-2}\,.$
- Reduced action

$$I_{red}[A, f, a, \phi, \varphi] = \frac{1}{16\pi G} \int d^D x \left(L_{EH} + L_{\phi\mathcal{A}} \right), \qquad (23)$$

where

$$L_{EH} = -e^A \left[\left((r^{D-2})' f \right)' + 2r^{D-2}\Lambda \right] - \left[r^{D-2} e^A f' + 2r^{D-2} (e^A)' f \right]',$$

$$L_{\phi\mathcal{A}} = -\frac{1}{2} r^{D-2} e^A f \phi'^2 + \frac{1}{2} r^{D-2} e^{-A} e^{\lambda\phi} a'^2\,.$$

Scaling symmetry

- Scaling transformation

$$\delta_\sigma f = \sigma(2f - r\,f')\,, \quad \delta_\sigma e^A = \sigma\big(-(D-1) - r\,A'\big)e^A\,, \tag{24}$$

$$\delta_\sigma \phi = -\sigma r\,\phi'\,, \quad \delta_\sigma a = \sigma\big(-(D-2)a - r\,a'\big)\,. \tag{25}$$

- Noether charge, $C = C(r)$

$$8GC = -\,r^{D-3}e^A\Big[(D-2)\Big(2f - rf'\Big) - r^2 f\phi'^2\Big] - (D-2)\,q\,a\,, \tag{26}$$

where

$$r^{D-2}e^{-A}e^{\lambda\phi}a' = \,q\,. \tag{27}$$

The Smarr relation

- At the horizon from direct computation and $C = C(r_H)$

$$(D-2)T_H\mathcal{S}_{BHW} = \frac{1}{2\pi}\left(C - C_0\right)\,, \qquad C_0 \equiv -\frac{D-2}{8G}qa(r_H)\,.$$

- At the asymptotic from direct computation

$$\delta M_{ADT} = \frac{1}{8\pi G}\int_{r\to\infty} dx_{\mu\nu}\Big(\delta K^{\mu\nu}(\xi_T) - 2\xi_T^{[\mu}\Theta^{\nu]}(\delta\Psi)\Big)$$
$$= \frac{1}{16\pi G}\Big[-(D-2)r^{D-3}e^{A}\delta f - a\,\delta q - r^{D-2}e^{A}f(\phi'\,\delta\phi)\Big]_{r\to\infty}$$

- ‘on-shell’ scaling transformation to get finite expression

$$\hat{\delta}_\sigma f = \sigma(2f - r\,f')\,, \quad \hat{\delta}_\sigma e^A = \sigma(z-1-r\,A')e^A\,,$$
$$\hat{\delta}_\sigma \phi = -\sigma r\,\phi'\,, \quad \hat{\delta}_\sigma a = \sigma(za - r\,a')\,.$$

The Smarr relation

- unwanted transformation

$$\hat{\delta}_\sigma \mu = -\sigma\sqrt{2(D-2)(z-1)}\,\mu\,. \tag{28}$$

- Mass expression

$$\delta M_{ADT} = \hat{\delta}_\sigma M_{ADT} + \hat{\delta}_\mu M_{ADT} = \sigma(D+z-2)M_{ADT}\,. \tag{29}$$

with

$$\hat{\delta}_\sigma M_{ADT} = \frac{\sigma}{2\pi}C\,. \tag{30}$$

- Smarr relation

$$\frac{1}{2\pi}(C-C_0) = (D+z-2)M_{ADT} = (D-2)T_H\mathcal{S}_{BHW}\,. \tag{31}$$

Conclusion

- We checked the Lifshitz Smarr relation for the NMG for various examples.

$$T_H \mathcal{S} = (1+z)M$$

- We obtained the Lifshitz Smarr relation for the specitic EMD gravity, also with scalar hair.

$$T_H \mathcal{S} = \frac{D+z-2}{D-2} M$$

- For 'rotating' BH with one Killing vector. [arXiv: 1508.06484]

$$M_\infty = \frac{1}{2} T_H \mathcal{S}_H + \Omega_H J_\infty - \frac{1}{32\pi G} \int drdy \left[\frac{\delta I_{red}}{\delta \dot{\Psi}} \dot{\Psi} \right]$$
$$dM_\infty = T_H d\mathcal{S}_H + \Omega_H dJ_\infty$$